Analytic Moments for GARCH Processes

Carol Alexander
ICMA Centre, Henley Business School at Reading

Emese Lazar
ICMA Centre, Henley Business School at Reading

Silvia Stanescu
Kent Business School, University of Kent

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ICMA Centre • The University of Reading
Whiteknights • PO Box 242 • Reading RG6 6BA • UK
Tel: +44 (0)1183 788239 • Fax: +44 (0)1189 314741
Web: www.icmacentre.rdg.ac.uk
Director: Professor John Board, Chair in Finance

ABSTRACT

Conditional returns distributions generated by a GARCH process, which are important for many problems in market risk assessment and portfolio optimization, are typically generated via simulation. This paper extends previous research on analytic moments of GARCH returns distributions in several ways: we consider a general GARCH model – the GJR specification with a generic innovation distribution; we derive analytic expressions for the first four conditional moments of the forward return, of the forward variance, of the aggregated return and of the aggregated variance – corresponding moments for some specific GARCH models largely used in practice are recovered as special cases; we derive the limits of these moments as the time horizon increases, establishing regularity conditions for the moments of aggregated returns to converge to normal moments; and we demonstrate empirically that some excellent approximate predictive distributions can be obtained from these analytic moments, thus precluding the need for time-consuming simulations.

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Carol Alexander
Chair of Risk Management,
ICMA Centre, Henley Business School at Reading,
Reading, RG6 6BA, UK.
Email: c.alexander@icmacentre.ac.uk

Emese Lazar
Lecturer in Finance,
ICMA Centre, Henley Business School at Reading,
Reading, RG6 6BA, UK.
Email: e.lazar@icmacentre.ac.uk

Silvia Stanescu
Lecturer in Finance,
Kent Business School, University of Kent,
Canterbury, Kent, CT2 7PE, UK.
Email: s.stanescu@kent.ac.uk

1 Introduction

Forward-looking physical return distributions have attracted a vast academic research literature because they have a great variety of financial applications to market risk assessment and portfolio optimization techniques. Since Mandelbrot (1963) and Fama (1965) it has been recognized that time series of asset returns are not well described by normal, independent processes. Typically, their conditional distributions are non-normal and they exhibit volatility clustering, so are not independent. Hence, forecasts of entire returns distributions – not only of their first two moments – are required in numerous financial applications.

The family of generalized autoregressive conditional heteroskedasticity (GARCH) models has proved highly successful in capturing (at least partially) the salient empirical features of both conditional and unconditional returns distributions. Following the pioneering work of Engle (1982), Bollerslev (1986) and Taylor (1986) numerous alternative specifications for GARCH processes have been proposed. In many financial markets, especially equities and commodities, the GARCH conditional variance equation captures the asymmetric response of volatility to innovations with different signs. Well-known asymmetric GARCH models include the EGARCH model of Nelson (1991), the AGARCH model of Engle (1990), Engle and Ng (1993) and Zakoian (1994), the NGARCH model also proposed by Engle and Ng (1993), the QGARCH model of Engle (1990), Sentana (1991) and Campbell and Hentschel (1992) and the model of Glosten, Jagannathan and Runkle (1993), henceforth denoted GJR. Additionally, GARCH models with non-normal innovation distributions have been developed by Bollerslev (1987), Nelson (1991), Haas, Mittnik, Paolella (2004) and many others.

The performance of various GARCH models has been emprically assessed by Andersen and Bollerslev (1998), Marcucci (2005) and many others.[1] Virtually all this literature refers to the accuracy of forward or aggregated returns distributions when a point GARCH variance forecast is used. However, only the one-step-ahead GARCH variance forecast is deterministic: due to the uncertainty about future returns, the forward returns variances, and variances of aggregated future returns, are stochastic. So a point GARCH variance forecast represents only an expected value of the GARCH variance, under its distribution. Until now, the only papers to have examined this distribution are

[1] Also, Andersen, Bollerslev and Diebold (2009) give a broad overview of volatility modelling procedures, focusing on the GARCH methodology and Bauwens, Laurent and Rombouts (2006) review some important contributions to the multivariate ARCH literature.

Ishida and Engle (2002), who derived the conditional variance of the forward conditional variance for a symmetric GARCH(1,1) model with symmetric innovations, and Christoffersen *et al.* (2010) who derived the second moment of the two-step-ahead forward variance for eight GARCH processes with affine vs. non-affine and conditionally-normal vs. conditionally GED alternatives.

By contrast, there has been considerable research on the unconditional moments of returns generated by GARCH processes.[2] However, since returns are not identically distributed, the conditional moments and their dynamics are most important for many financial applications. Knowledge of the dynamics of the conditional mean and variance is sufficient only when conditional distributions are normal: more generally, the dynamics of higher order conditional moments are needed. Hence, most recent research focuses on the first four conditional moments of forward and aggregated returns for some specific GARCH processes.

Duan *et al.* (1999) derived expressions for the first four conditional moments of the aggregated returns generated by the NGARCH model under the risk-neutral probability measure, and Duan *et al.* (2006) extended these results to the risk-neutral moments of aggregated returns under normal GJR and normal EGARCH processes. Wong and So (2003) derived an expression for the variance of aggregated QGARCH returns and, under the additional assumption that the innovation is symmetric, expressions for the third and fourth order conditional moments of aggregate returns.[3] One paper has investigated the limiting distributions of GARCH aggregated returns as the aggregation horizon increases indefinitely. For a generic symmetric GARCH(1,1) process, Breuer and Jandacka (2007) derived the limit of the variance and kurtosis of forward and aggregated returns. Both the forward skewness and the skewness of aggregated returns is zero by construction in their framework.[4]

The purpose of our paper is to extend previous research on the conditional moments of both returns and variances, both forward and aggregated, based on a unified framework in which many of the results cited above may be derived as special cases. Assuming a GJR specification and assuming

[2]See Engle (1982), Nemec (1985), Milhoj (1985), Bollerslev (1986), He and Terasvirta (1999a, 1999b), Karanasos (1999, 2001), He, Terasvirta and Malmsten (2002), Demos (2002), Ling and McAleer (2002a, 2002b), Karanasos and Kim (2003), Bai, Russell and Tiao (2003) and Karanasos, Psaradakis and Sola (2004).

[3]Of some relation to this research is the paper by Christoffersen *et al.* (2008), who extended the work of Heston and Nandi (2000) to propose a new two-component volatility model for European options valuation, deriving the returns distribution moment generating function under certain (affine and conditionally normal) GARCH specifications.

[4]As opposed to this, we have a generic, asymmetric GARCH process and also allow innovations to be conditionally skewed; these generalizations allow us to model both forward and aggregated skewness.

a generic conditional distribution that can accommodate skewness and kurtosis in the innovations, we derive formulae for the first four conditional moments of forward and aggregated returns and the first four conditional moments of forward and aggregated variances and we also derive the limits of these moments as the returns horizon increases.[5] A simulation experiment compares approximate predictive returns distributions, derived from the conditional moments of returns, to the distributions that are generated by Monte Carlo simulation. If our moment formulae can be successfully applied to generate accurate distributions the need for time-consuming simulations may be precluded. Various GJR models are estimated on daily rolling windows of data on an equity index (S&P 500), foreign exchange rate (Euro/dollar) and an interest rate (3-month Treasury bill). Then we perform a thorough and comprehensive analysis of the fit between the approximate and simulated predictive returns distributions.

The formulae for the moments are presented in Section 2 and those for the limits in Section 3. The proofs are lengthy and are detailed in separate appendices. Also in the appendices we outline the results for the moments (and limits of moments) for important special cases of the generic model. In section 4 the moments are used to derive approximate predictive distributions for the forward returns and statistical tests determine the goodness-of-fit between these distributions and those that are generated via simulation; Section 5 summarizes and concludes.

2 Moments of Generic GJR Returns and Variances

We present analytic expressions for the first four conditional moments of both forward one-period and future aggregated (also called cumulative) returns and variances for the GJR model with a generic innovation distribution having zero mean, unit variance and finite higher moments. We assume that the one-period log return $r_t = \log\left(\frac{P_{t+1}}{P_t}\right)$ on a financial asset with market price P_t follows a stationary processes with no significant autocorrelation. If returns exhibit autocorrelation we de-autocorrelate the data before estimating the GARCH parameters. The mathematical specification of the generic GJR model is:

[5] The GJR model encompasses many of the specific models that have been assumed in the previous research cited above but it does not encompass the AGARCH model. The first four moments of the forward and aggregated returns generated by an AGARCH process have, however, been derived and are available from the authors on request.

$$r_t = \mu + \varepsilon_t, \quad \varepsilon_t = z_t h_t^{1/2}, \quad z_t \sim D(0,1), \quad h_t = \omega + \alpha\, \varepsilon_{t-1}^2 + \lambda\, \varepsilon_{t-1}^2 I_{t-1}^- + \beta\, h_{t-1}, \tag{1}$$

where I_t^- is an indicator function which equals 1 if $\varepsilon_t < 0$ and zero otherwise, z_t and h_t are independent and $D(0,1)$ is a generic conditional distribution with zero mean, unit variance, constant skewness τ_z and kurtosis κ_z, and with constant higher moments $\mu_z^{(i)} = E\left(z_t^i\right)$ for any $i > 4,\ i \in N$.[6]

The aggregated return over n consecutive time periods is denoted $R_{tn} = \sum_{s=1}^{n} r_{t+s}$, and for the conditional un-centred and centred moments of the forward and aggregated returns and variances the following notation is used:

$$\tilde{\mu}_{x,s}^{(i)} = E_t\left(x_{t+s}^i\right), \quad \mu_{x,s}^{(i)} = E_t\left(\left(x_{t+s} - \tilde{\mu}_{x,s}^{(1)}\right)^i\right)$$
$$\tilde{M}_{x,n}^{(i)} = E_t\left[\left(\sum_{s=1}^{n} x_{t+s}\right)^i\right], \quad M_{x,n}^{(i)} = E_t\left(\left(\sum_{s=1}^{n}\left(x_{t+s} - \tilde{\mu}_{x,s}^{(1)}\right)\right)^i\right)$$

for $x = r$ and h in turn, and for $s = 1, 2,..., n$ and $i = 1, 2, 3, 4$. Thus, the skewness and kurtosis of the forward (return or variance) distributions are:

$$\tau_{x,s} = \mu_{x,s}^{(3)}\left(\mu_{x,s}^{(2)}\right)^{-3/2} \quad \text{and} \quad \kappa_{x,s} = \mu_{x,s}^{(4)}\left(\mu_{x,s}^{(2)}\right)^{-2}$$

and the skewness and kurtosis of the aggregated return or variance distributions are:

$$\mathrm{T}_{x,n} = M_{x,n}^{(3)}\left(M_{x,n}^{(2)}\right)^{-3/2} \quad \text{and} \quad \mathrm{K}_{x,n} = M_{x,n}^{(4)}\left(M_{x,n}^{(2)}\right)^{-2}.$$

We start with the un-centred moments, namely:

$$E_t\left(x_{t+s}^i\right) \quad \text{and} \quad E_t\left[\left(\sum_{s=1}^{n} x_{t+s}\right)^i\right]$$

again, for $x = r$ and h in turn, and for $s = 1, 2,..., n$ and $i = 1, 2, 3, 4$. Subsequently, we obtain the centred and standardized moments of the GJR process with a generic innovation distribution. The derivations rely on the observation that, although $E_t\left(h_{t+1}\right) = h_{t+1}$ (i.e. $V_t\left(h_{t+1}\right) = 0$) both $\left\{h_{t+s}\left|\Omega_t : s \in N\backslash\{0,1\}\right.\right\}$ and $\left\{\sum_{s=1}^{n} h_{t+s}\left|\Omega_t : n \in N\backslash\{0,1\}\right.\right\}$ are random. Moreover, both $\left\{r_{t+s}\left|\Omega_t : s \in N\backslash\{0\}\right.\right\}$ and $\left\{R_{tn}\left|\Omega_t : n \in N\backslash\{0\}\right.\right\}$ are random and have distributions that can also be approximated using moments that we derive.

[6] To be more precise, we have:

$$\mu_z^{(i)} = E\left(z_t^i\right) = E_t\left(z_t^i\right) = E_t\left(z_t - E_t\left(z_t\right)\right)^i = E_t\left(z_t - E_t\left(z_t\right)\right)^i E_t\left(z_t^2\right)^{-i/2}$$

since un-centred, centred and standardized moments are all equal for a zero mean, unit variance distribution. Also, since the z process is i.i.d., conditional and unconditional moments of z are also identical. Actually, for the fourth conditional moment of returns to exist we need the first four moments of z to be finite, while we require up to the eighth moment of z to be finite in order to have a finite fourth conditional moment of future variances.

The following results and notation will be used:

1. $\varphi = \alpha + \lambda F_0 + \beta$, with F_0 being the distribution function for $D(0,1)$ evaluated at zero;
2. $\bar{h} = \omega(1-\varphi)^{-1}$; if $\varphi \in (0,1)$, then $\bar{h}$ is the steady-state variance towards which the conditional variance mean reverts;
3. $$\tilde{\mu}_{h,s}^{(2)} = c_1 + \left(h_{t+1}^2 - c_3\right)\gamma^{s-1} + c_2\varphi^{s-1}, \tag{2}$$
   where $\gamma = \varphi^2 + (\kappa_z - 1)(\alpha + \lambda F_0)^2 + \kappa_z \lambda^2 F_0 (1 - F_0)$, $c_1 = \left(\omega^2 + 2\omega\varphi\bar{h}\right)(1-\gamma)^{-1}$, $c_2 = 2\omega\varphi\left(h_{t+1} - \bar{h}\right)(\varphi - \gamma)^{-1}$ and $c_3 = c_1 + c_2$.
4. $$E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right) = \varphi^{u-1}\left(\alpha\tau_z + \lambda \int\limits_{x=-\infty}^{0} x^3 f(x)\, dx\right) E_t\left(h_{t+s}^{3/2}\right), \tag{3}$$
   where f is the density function of $D(0,1)$.

We also need expressions for $E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^3\right)$, $E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right)$, and $E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}^2\right)$ in terms of the model parameters, but since these are lengthy they are only stated in the Appendix A.1, with the proof of the following result:

Theorem 1. Moments of Forward and Aggregated Returns

The conditional moments of forward one-period returns of model (1) are:

$$\tilde{\mu}_{r,s}^{(1)} = \mu, \quad \mu_{r,s}^{(2)} = \tilde{\mu}_{h,s}^{(1)} = \bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right),$$
$$\tau_{r,s} = \tau_z E_t\left(h_{t+s}^{3/2}\right)\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-3/2} \simeq \tau_z\left(\tfrac{5}{8} + \tfrac{3}{8}\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-2}\right),$$
$$\kappa_{r,s} = \kappa_z \tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-2}.$$

The conditional moments of the aggregated returns of model (1) are:

$$\tilde{M}_{r,n}^{(1)} = n\mu, \quad M_{r,n}^{(2)} = n\bar{h} + (1-\varphi)^{-1}(1-\varphi^n)\left(h_{t+1} - \bar{h}\right),$$
$$\mathrm{T}_{r,n} \simeq \left(\tau_z \sum_{s=1}^{n}\left(\tfrac{5}{8}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3/2} + \tfrac{3}{8}\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-1/2}\right) + 3\sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right)\right)\left(M_{r,n}^{(2)}\right)^{-3/2},$$
$$\mathrm{K}_{r,n} = \left(\begin{array}{l} \kappa_z \sum\limits_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)} + \sum\limits_{s=1}^{n}\sum\limits_{u=1}^{n-s}\left(4E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^3\right) + 6E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right)\right) \\ +12\sum\limits_{s=1}^{n}\sum\limits_{u=1}^{n-s}\sum\limits_{v=1}^{n-s-u} E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}^2\right) \end{array}\right)\left(M_{r,n}^{(2)}\right)^{-2}.$$

The first conditional moments $\tilde{\mu}_{r,s}^{(1)}$ and $\tilde{M}_{r,n}^{(1)}$ simply state that with a constant conditional mean equation the time t conditional expectation of the s-step-ahead one-period return is equal to the constant conditional mean, whereas the expected return aggregated over n periods scales with time. The second moment of the forward return $\mu_{r,s}^{(2)}$ shows that the conditional expectation of the s-step-ahead variance is equal to the steady state variance $\bar{h}$, plus an exponentially decreasing correction term to account for the distance between the one-step-ahead variance h_{t+1} and the steady state variance $\bar{h}$. Because we assume that the returns are not autocorrelated, the variance of aggregated returns over n time periods $M_{r,n}^{(2)}$ is simply equal to the sum of the s-step-ahead variances for s = 1, 2, ..., n.

The expressions for the forward and aggregated skewness are obtained using a second order Taylor series expansion for $E_t\left(h_{t+s}^{3/2}\right)$, as detailed in Appendix A.2. It is easily observed that if the innovation is symmetric ($\tau_z = 0$) then the forward returns distribution is also symmetric. By contrast, considering the expression for $E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right)$ in (3), cumulative returns have an independent source of skewness in addition to that of the innovations τ_z, due to the asymmetric response parameter λ in the conditional variance equation. As a result, aggregated returns can exhibit skewness even if the innovation is symmetric.

For $s = 1$, the forward kurtosis equals the kurtosis of the innovation process. But for $s > 1$, the forward excess kurtosis can be non-zero even when the innovation has zero excess kurtosis, due to the uncertainty in forward variance. The conditional variance of the conditional variance varies with s and, as it must be positive, the forward kurtosis will be greater than the kurtosis of the innovation, whenever $s > 1$, and will itself be time-varying. The net effect of uncertainty in variance is a greater weight in the tails of forward one-period returns. Also, the time-varying conditional variance of the conditional variance introduces dynamics in the higher moments of the forward returns.

The moments of variances require the following results (proved in Appendix A.2):

$$\tilde{\mu}_{h,s}^{(3)} = \sum_{i=0}^{s-2} c_4^i \left(\omega^3 + 3\omega^2\varphi\tilde{\mu}_{h,s-i-1}^{(1)} + 3\omega\gamma\tilde{\mu}_{h,s-i-1}^{(2)}\right) + c_4^{s-1}h_{t+1}^3,$$

with

$$\begin{aligned} c_4 &= \mu_z^{(6)}\left(\alpha^3 + 3\alpha\lambda\left(\alpha+\lambda\right)F_0 + \lambda^3 F_0\right) + 3\beta\gamma - \beta^2\left(2\beta + 3\left(\alpha + \lambda F_0\right)\right) \\ \tilde{\mu}_{h,s}^{(4)} &= \sum_{j=0}^{s-2} c_7^j \left(\omega^4 + 4\omega^3\varphi\tilde{\mu}_{h,s-j-1}^{(1)} + c_5\tilde{\mu}_{h,s-j-1}^{(2)} + c_6\tilde{\mu}_{h,s-j-1}^{(3)}\right) + c_7^{s-1}h_{t+1}^4, \end{aligned} \quad (4)$$

with $c_5 = 6\omega^2\gamma$, $c_6 = 4\omega c_4$, and

$$\begin{aligned} c_7 &= \mu_z^{(8)}\left(\alpha^4 + F_0\left(\lambda^4 + 4\left(\alpha^3\lambda + \alpha\lambda^3\right) + 6\alpha^2\lambda^2\right)\right) + \beta^4 \\ &+ 4\left[\mu_z^{(6)}\beta\left(\alpha^3 + F_0\left(\lambda^3 + 3\left(\alpha^2\lambda + \alpha\lambda^2\right)\right)\right) + \beta^3\left(\alpha + \lambda F_0\right)\right] + 6\kappa_z\beta^2\left(\alpha^2 + \lambda^2 F_0 + 2\alpha\lambda F_0\right). \end{aligned}$$

Expressions for $\tilde{\mu}_{h,suv}^{(i,j,k)}$, with $i, j, k \in \{0,1,2\}$ are also required but since most are rather lengthy they are only stated in Appendix A.2, with the proof of the following result:

Theorem 2. Moments of Forward and Aggregated Variances

The conditional moments of forward one-period variances of model (1) are:

$$\tilde{\mu}_{h,s}^{(1)} = \bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right), \quad \mu_{h,s}^{(2)} = \tilde{\mu}_{h,s}^{(2)} - \left(\tilde{\mu}_{h,s}^{(1)}\right)^2,$$

$$\tau_{h,s} = \left[\tilde{\mu}_{h,s}^{(3)} - 3\tilde{\mu}_{h,s}^{(2)}\tilde{\mu}_{h,s}^{(1)} + 2\left(\tilde{\mu}_{h,s}^{(1)}\right)^3\right]\left(\tilde{\mu}_{h,s}^{(2)} - \tilde{\mu}_{h,s}^{(1)}\right)^{-3/2},$$

$$\kappa_{h,s} = \left(\tilde{\mu}_{h,s}^{(4)} - 4\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s}^{(3)} + 6\left(\tilde{\mu}_{h,s}^{(1)}\right)^2\tilde{\mu}_{h,s}^{(2)} - 3\left(\tilde{\mu}_{h,s}^{(1)}\right)^4\right)\left(\tilde{\mu}_{h,s}^{(2)} - \left(\tilde{\mu}_{h,s}^{(1)}\right)^2\right)^{-2}.$$

The conditional moments of the aggregated future variances of model (1) are:

$$\tilde{M}_{h,n}^{(1)} = n\bar{h} + \left(h_{t+1} - \bar{h}\right)(1-\varphi)^{-1}\left(1 - \varphi^n\right),$$

$$M_{h,n}^{(2)} = \sum_{s=1}^{n}\left(\tilde{\mu}_{h,s}^{(2)} - \left(\tilde{\mu}_{h,s}^{(1)}\right)^2\right) + 2\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\tilde{\mu}_{h,su}^{(1,1)} - \tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s+u}^{(1)}\right),$$

$$\mathrm{T}_{h,n} = M_{h,n}^{(3)}\left(M_{h,n}^{(2)}\right)^{-3/2},$$

$$M_{h,n}^{(3)} = \sum_{s=1}^{n}\left(\tilde{\mu}_{h,s}^{(3)} - 3\tilde{\mu}_{h,s}^{(2)}\tilde{\mu}_{h,s}^{(1)} + 2\left(\tilde{\mu}_{h,s}^{(1)}\right)^3\right) + 3\sum_{s=1}^{n}\sum_{u=1}^{n-s}A_{h,s,u} + 6\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}B_{h,s,u,v},$$

$$A_{h,s,u} = \tilde{\mu}_{h,su}^{(2,1)} + \tilde{\mu}_{h,su}^{(1,2)} + 2\left(\tilde{\mu}_{h,s}^{(1)} + \tilde{\mu}_{h,s+u}^{(1)}\right)\left(\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s+u}^{(1)} - \tilde{\mu}_{h,su}^{(1,1)}\right) - \tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s+u}^{(2)} - \tilde{\mu}_{h,s+u}^{(1)}\tilde{\mu}_{h,s}^{(2)},$$

$$B_{h,s,u,v} = \tilde{\mu}_{h,suv}^{(1,1,1)} - \tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,(s+u)v}^{(1,1)} - \tilde{\mu}_{h,(s+u)}^{(1)}\tilde{\mu}_{h,s(u+v)}^{(1,1)} - \tilde{\mu}_{h,(s+u+v)}^{(1)}\tilde{\mu}_{h,su}^{(1,1)} + 2\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,(s+u)}^{(1)}\tilde{\mu}_{h,(s+u+v)}^{(1)},$$

$$\mathrm{K}_{h,n} = M_{h,n}^{(4)}\left(M_{h,n}^{(2)}\right)^{-2}$$

$$M_{h,n}^{(4)} = \sum_{s=1}^{n}\mu_{h,s}^{(4)} + \sum_{s=1}^{n}\sum_{u=1}^{n-s}C_{h,s,u} + 12\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}D_{h,s,u,v} + 24\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\sum_{w=1}^{n-s-u-v}E_{h,s,u,v,w},$$

$$\begin{aligned} C_{h,s,u} = 4 &\left(\begin{array}{l} \tilde{\mu}_{h,su}^{(3,1)} + \tilde{\mu}_{h,su}^{(1,3)} - 3\left(\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,su}^{(2,1)} + \tilde{\mu}_{h,s+u}^{(1)}\tilde{\mu}_{h,su}^{(1,2)}\right) - \left(\tilde{\mu}_{h,s+u}^{(1)}\tilde{\mu}_{h,s}^{(3)} + \tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s+u}^{(3)}\right) \\ +3\tilde{\mu}_{h,s}^{(1)}\left(\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,su}^{(1,1)} + \tilde{\mu}_{h,s+u}^{(1)}\tilde{\mu}_{h,s}^{(2)} - \left(\tilde{\mu}_{h,s}^{(1)}\right)^2\tilde{\mu}_{h,s+u}^{(1)}\right) \\ +3\tilde{\mu}_{h,s+u}^{(1)}\left(\tilde{\mu}_{h,s+u}^{(1)}\tilde{\mu}_{h,su}^{(1,1)} + \tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s+u}^{(2)} - \tilde{\mu}_{h,s}^{(1)}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^2\right) \end{array}\right) \\ +6 &\left(\begin{array}{l} \tilde{\mu}_{h,su}^{(2,2)} - 2\left(\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,su}^{(1,2)} + \tilde{\mu}_{h,s+u}^{(1)}\tilde{\mu}_{h,su}^{(2,1)}\right) + \left(\tilde{\mu}_{h,s}^{(1)}\right)^2\tilde{\mu}_{h,s+u}^{(2)} \\ +\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^2\tilde{\mu}_{h,s}^{(2)} + 4\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s+u}^{(1)}\tilde{\mu}_{h,su}^{(1,1)} - 3\left(\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s+u}^{(1)}\right)^2 \end{array}\right), \end{aligned}$$

$$
\begin{aligned}
D_{h,s,u,v} = {} & \tilde{\mu}^{(2,1,1)}_{h,suv} - 2\tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1,1,1)}_{h,suv} - \tilde{\mu}^{(1)}_{h,s+u}\tilde{\mu}^{(2,1)}_{h,s(u+v)} - \tilde{\mu}^{(1)}_{h,s+u+v}\tilde{\mu}^{(2,1)}_{h,su} + \left(\tilde{\mu}^{(1)}_{h,s}\right)^2\tilde{\mu}^{(1,1)}_{h,(s+u)v} \\
& +2\tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1)}_{h,s+u}\tilde{\mu}^{(1,1)}_{h,s(u+v)} + 2\tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1)}_{h,s+u+v}\tilde{\mu}^{(1,1)}_{h,su} + \tilde{\mu}^{(1)}_{h,s+u}\tilde{\mu}^{(1)}_{h,s+u+v}\tilde{\mu}^{(2)}_{h,s} \\
& -3\left(\tilde{\mu}^{(1)}_{h,s}\right)^2\tilde{\mu}^{(1)}_{h,s+u}\tilde{\mu}^{(1)}_{h,s+u+v} + \tilde{\mu}^{(1,2,1)}_{h,suv} - 2\tilde{\mu}^{(1)}_{h,s+u}\tilde{\mu}^{(1,1,1)}_{h,suv} - \tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(2,1)}_{h,(s+u)v} - \tilde{\mu}^{(1)}_{h,s+u+v}\tilde{\mu}^{(1,2)}_{h,su} \\
& +\left(\tilde{\mu}^{(1)}_{h,s+u}\right)^2\tilde{\mu}^{(1,1)}_{h,s(u+v)} + 2\tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1)}_{h,s+u}\tilde{\mu}^{(1,1)}_{h,(s+u)v} + 2\tilde{\mu}^{(1)}_{h,s+u}\tilde{\mu}^{(1)}_{h,s+u+v}\tilde{\mu}^{(1,1)}_{h,su} \\
& +\tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1)}_{h,s+u+v}\tilde{\mu}^{(2)}_{h,s+u} - 3\tilde{\mu}^{(1)}_{h,s}\left(\tilde{\mu}^{(1)}_{h,s+u}\right)^2\tilde{\mu}^{(1)}_{h,s+u+v} + \tilde{\mu}^{(1,1,2)}_{h,suv} - 2\tilde{\mu}^{(1)}_{h,s+u+v}\tilde{\mu}^{(1,1,1)}_{h,suv} \\
& -\tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1,2)}_{h,(s+u)v} - \tilde{\mu}^{(1)}_{h,s+u}\tilde{\mu}^{(1,2)}_{h,s(u+v)} + \left(\tilde{\mu}^{(1)}_{h,s+u+v}\right)^2\tilde{\mu}^{(1,1)}_{h,su} + 2\tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1)}_{h,s+u+v}\tilde{\mu}^{(1,1)}_{h,(s+u)v} \\
& +2\tilde{\mu}^{(1)}_{h,s+u}\tilde{\mu}^{(1)}_{h,s+u+v}\tilde{\mu}^{(1,1)}_{h,s(u+v)} + \tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1)}_{h,s+u}\tilde{\mu}^{(2)}_{h,s+u+v} - 3\tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1)}_{h,s+u}\left(\tilde{\mu}^{(1)}_{h,s+u+v}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
E_{h,s,u,v,w} = {} & \tilde{\mu}^{(1,1,1,1)}_{h,suvw} - \tilde{\mu}^{(1)}_{h,s+u+v}\tilde{\mu}^{(1,1,1)}_{h,su(v+w)} - \tilde{\mu}^{(1)}_{h,s+u+v+w}\tilde{\mu}^{(1,1,1)}_{h,suv} + \tilde{\mu}^{(1)}_{h,s+u+v}\tilde{\mu}^{(1)}_{h,s+u+v+w}\tilde{\mu}^{(1,1)}_{h,su} - \tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1,1,1)}_{h,(s+u)vw} \\
& +\tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1)}_{h,s+u+v}\tilde{\mu}^{(1,1)}_{h,(s+u)(v+w)} + \tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1)}_{h,s+u+v+w}\tilde{\mu}^{(1,1)}_{h,(s+u)v} - \tilde{\mu}^{(1)}_{h,s+u}\tilde{\mu}^{(1,1,1)}_{h,(s+u)vw} + \tilde{\mu}^{(1)}_{h,s+u}\tilde{\mu}^{(1)}_{h,s+u+v}\tilde{\mu}^{(1,1)}_{h,s(u+w+v)} \\
& +\tilde{\mu}^{(1)}_{h,s+u}\tilde{\mu}^{(1)}_{h,s+u+v+w}\tilde{\mu}^{(1,1)}_{h,s(u+v)} + \tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1)}_{h,s+u}\tilde{\mu}^{(1,1)}_{h,(s+u+v)w} - 3\tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1)}_{h,s+u}\tilde{\mu}^{(1)}_{h,s+u+v}\tilde{\mu}^{(1)}_{h,s+u+v+w}
\end{aligned}
$$

It has been argued, for instance in Ishida and Engle (2002), that the conditional variance of the conditional variance grows faster than linearly with the current variance. Our formula for $\mu^{(2)}_{h,s}$ shows that, in the context of model (1), it grows quadratically, as from this it easily follows that the conditional variance of the forward conditional variance is a quadratic function of the current variance h_{t+1}. Hence, the uncertainty around the point variance forecast increases much more than linearly when variance levels are high, much reducing the reliability of the point forecast. This highlights the importance of our analytic formulae for the higher moments of GARCH variances. [7]

3 LIMITS OF THE MOMENTS

The convergence behaviour of the conditional moments as the time horizon increases is of interest both theoretically and empirically. The following result, proved in Appendix A.3, summarizes the limits of the moments of returns (the limits of the forward and aggregated mean are trivial and immediate and are thus excluded).

[7] Intuitively, we expect distributions of forward variances to be positively skewed, since jumps in variance are usually positive rather than negative. In an empirical implementation of the moments formulae derived in this section we find that the skewness of forward variance is indeed positive, for all horizons and all three samples considered; we also find that the excess kurtosis of variance is always positive. These empirical results are excluded from this paper for reasons of space, but can be obtain from the authors on request.

Theorem 3. Limiting Behaviour of Moments of Forward and Aggregated Returns

Suppose $0 < \varphi < 1$ and $\varphi \neq \gamma$. The limiting behaviour of the moments of the forward one-period and aggregated returns when we increase the time horizon is:[8]

$$\lim_{s\to\infty} \mu_{r,s}^{(2)} = \bar{h}, \tag{5}$$

$$\lim_{s\to\infty} \tau_{r,s} = \begin{cases} \tau_z \left(\frac{5}{8} + \frac{3}{8}\left(\omega^2 + 2\omega\varphi\bar{h}\right)(1-\gamma)^{-1}\left(\bar{h}\right)^{-2}\right) & \text{if } \gamma \in (0,1) \\ \operatorname{sgn}(\tau_z)\,\infty & \text{if } \gamma \in [1,\infty), \end{cases} \tag{6}$$

$$\lim_{s\to\infty} \kappa_{r,s} = \begin{cases} \kappa_z \left(\omega^2 + 2\omega\varphi\bar{h}\right)(1-\gamma)^{-1}\left(\bar{h}\right)^{-2} & \text{if } \gamma \in (0,1) \\ \infty & \text{if } \gamma \in [1,\infty), \end{cases} \tag{7}$$

$$\lim_{n\to\infty} \frac{M_{r,n}^{(2)}}{n} = \bar{h}, \tag{8}$$

$$\lim_{n\to\infty} \mathrm{T}_{r,n} = \begin{cases} 0 & \text{if } \gamma \in (0,1) \\ \operatorname{sgn}\left(\tau_z\left(\alpha + \frac{\gamma-\varphi}{3}\right) + \lambda \int\limits_{x=-\infty}^{0} x^3 f(x)\,dx\right)\infty & \text{if } \gamma \in [1,\infty), \end{cases} \tag{9}$$

$$\lim_{n\to\infty} \mathrm{K}_{r,n} = \begin{cases} 3 & \text{if } \gamma \in (0,1) \\ 3 + \frac{\kappa_z}{2}\left(1-\varphi^2\right)\left(1 + 6\left(\alpha + \lambda F_0 + \kappa_z^{-1}\beta\right)(1-\varphi)^{-1}\right) + \operatorname{sgn}\left(|\lambda| + |\tau_z|\right)\infty & \text{if } \gamma = 1 \\ \infty & \text{if } \gamma \in (1,\infty). \end{cases} \tag{10}$$

Hence, under suitable parameter conditions, the conditional moments of forward one-period returns converge to finite limits that are the unconditional counterparts of the respective conditional moments, and these parameter conditions are the necessary and sufficient conditions for the existence of the corresponding unconditional moments. Indeed, $\varphi \in (0,1)$ is a necessary and sufficient condition for the existence of the unconditional variance, as can be shown using Theorem 2.2 (and Example 2.1) in Ling and McAleer(2002b), and for $\varphi \in (0,1)$ a steady-state level of variance exists, i.e. $\exists\ h_0$ such that $E(h_t) = h_0$, for any $t \in N$. It is easy to show that, when it exists,

[8] $\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$ and we use the convention that $\operatorname{sgn}(0)\infty = 0$.

the unconditional variance h_0 is given by: $h_0 = \frac{\omega}{1-\varphi} = \bar{h}$.[9] Thus we obtain that for $\varphi \in (0,1)$, $\lim_{s\to\infty} \mu_{r,s}^{(2)} = \lim_{s\to\infty} E_t\left(h_{t+s}\right) = \bar{h} = E\left(h_{t+s}\right)$.

Again, using Theorem 2.2 from Ling and McAleer(2002b), a necessary and sufficient condition for the existence of the fourth unconditional moment is $\gamma \in (0,1)$. Again, this is the condition required in Theorem 3 for the fourth conditional moment to converge to a finite limit. It is easy to show that, when it exists, the fourth unconditional moment is given by:

$$E\left(\varepsilon_t^4\right) = \kappa_z E\left(h_t^2\right) = \kappa_z \frac{\omega^2 + 2\omega\varphi\bar{h}}{1-\gamma} = \kappa_z c_1$$

and, as a result, the unconditional kurtosis is given by the same expression as in (7) above, for $\gamma \in (0,1)$. A special case of this result is the unconditional kurtosis for a GARCH(1,1) process with symmetric innovations, derived by Ishida and Engle (2002).

The unconditional skewness is given by:

$$\frac{E\left(\varepsilon_t^3\right)}{\left[E\left(\varepsilon_t^2\right)\right]^{3/2}} = \frac{E\left(z_t^3 h_t^{3/2}\right)}{\left[E\left(h_t\right)\right]^{3/2}} = \tau_z \frac{E\left(h_t^{3/2}\right)}{\left[E\left(h_t\right)\right]^{3/2}}.$$

Since $E\left(h_t^{3/2}\right)$ cannot be computed analytically in this framework, we use a second order Taylor series expansion to approximate it. Thus, $E\left(h_t^{3/2}\right) \simeq \frac{5}{8}\left(E\left(h_t\right)\right)^{3/2} + \frac{3}{8}E\left(h_t^2\right)\left(E\left(h_t\right)\right)^{-1/2}$ and a resulting approximation of the unconditional skewness is the same as the expression given in (6) for $\gamma \in (0,1)$.

The classical central limit theorem does not apply here, because the variables are not independent. Nevertheless, the conditional moments of the aggregated returns converge to the corresponding moments of a normal distribution, provided that certain parameter conditions are met. Outside of the regularity conditions, the conditional skewness of aggregated returns diverges to $\pm\infty$ and the conditional kurtosis of aggregated returns diverges to $+\infty$. This is similar to a result of Diebold (1988) who shows that the unconditional distribution of the aggregated returns for a conditionally

[9] Applying the expectation operator on both sides of the equation for the GJR conditional variance and using that the indicator I_t^- and the even powers of the contemporaneous innovations ε_t^{2k}, where $k \in N$, are independent, we get: $E\left(h_t\right) = \omega + \alpha\, E\left(\varepsilon_{t-1}^2\right) + \lambda E\left(\varepsilon_{t-1}^2\right) F_0 + \beta\, E\left(h_t\right)$. Using that $E_{t-2}\left(\varepsilon_{t-1}^2\right) = h_{t-1}$ and the tower law of expectations, we can write: $E\left(h_t\right) = \omega + \left(\alpha\, + \lambda F_0 + \beta\right)\, E\left(h_t\right)$, which yields, $E\left(h_t\right) = \frac{\omega}{1-\varphi} = \bar{h}$.

normal AR-ARCH (*m*, *p*) process also converges to a normal distribution, under suitable parameter conditions.

Interestingly, identical convergence conditions apply for the moments of both forward and aggregated returns. Whenever the moments of forward returns converge to the unconditional moments, the aggregated moments converge to the corresponding moments of a normal distribution. Moreover, for a special case of the generic framework, namely for the normal GARCH(1,1) model with $\gamma = 1$, the limit of the kurtosis of forward returns is infinite whilst the kurtosis of aggregated returns converges to a constant value different from 3. In fact, this additional convergence case for $\gamma = 1$ is not specific to the normal GARCH(1,1): it applies to any GARCH(1,1) model with symmetric innovations. This result, for the symmetric special case, is in agreement with Breuer and Jandacka (2007) even though our proof is different from theirs.

Theorem 4. Limiting Behaviour of Moments for Forward and Aggregated Variances
Suppose $0 < \varphi < 1$ and $\gamma \neq \varphi$ (as above); additionally $c_4 \neq \gamma$ and $c_4 \neq \varphi$. The following results are derived in Appendix A.4:

a) The limit of the conditional variance of the forward conditional variance of model (1) is:

$$\lim_{s\to\infty} \mu_{h,s}^{(2)} = \begin{cases} \left(\left(\omega^2 + 2\omega\varphi\bar{h}\right)(1-\gamma)^{-1} - \bar{h}^2\right) & \text{if } \gamma \in (0,1) \\ \infty & \text{if } \gamma \in [1,\infty) \end{cases} \tag{11}$$

b) The limit of the conditional variance of the aggregated conditional variance (per unit of time) of model (1) is:

$$\lim_{n\to\infty} \frac{M_{h,n}^{(2)}}{n} = \begin{cases} \left(\left(\omega^2 + 2\omega\varphi\bar{h}\right)(1-\gamma)^{-1} - \bar{h}^2\right)\left(1 + 2\varphi(1-\varphi)^{-1}\right) & \text{if } \gamma \in (0,1) \\ \infty & \text{if } \gamma \in [1,\infty) \end{cases} \tag{12}$$

c) The limit of the conditional skewness of the forward conditional variance of model (1) is:

$$\lim_{s\to\infty} \tau_{h,s} = \begin{cases} M_1 & \text{if } \gamma \in (0,1) \text{ and } c_4 \in (0,1) \\ 0 & \text{if } \gamma \in [1,\infty) \text{ and } c_4 \in \left(0, \gamma^{3/2}\right) \\ M_2 & \text{if } \gamma \in (1,\infty) \text{ and } c_4 = \gamma^{3/2} \\ \infty & \text{if } \{\gamma \in (0,1),\ c_4 \in [1,\infty)\} \text{ or } \{\gamma = 1,\ c_4 \in (1,\infty)\} \\ & \quad \text{or } \left\{\gamma \in (1,\infty),\ c_4 \in \left(\gamma^{3/2}, \infty\right)\right\} \end{cases} \tag{13}$$

where M_1 and M_2 are given in Appendix A.4.

d) For $\gamma \in (0,1)$ the conditional skewness of the aggregated conditional variance of model (1) has limit:

$$\lim_{n\to\infty} \mathrm{T}_{h,n} = \begin{cases} 0 & \text{if } c_4 \in (0,1) \\ \infty & \text{if } c_4 \in (1,\infty) \\ \operatorname{sgn}(N)\infty & \text{if } c_4 = 1 \end{cases}, \tag{14}$$

where N is given in Appendix A.4 and c_4 was defined in (4).[10]

The returns process has no autocorrelation so the variance of aggregated returns is just the sum of the forward one-period variances. However, the variance process is autocorrelated. As a result the variances of the two processes have different limiting behaviour. The limit of the variance of aggregated returns per unit of time is equal to the limit of forward variance (i.e. $\lim_{n\to\infty} \frac{\mathrm{M}^{(2)}_{r,n}}{n} = \lim_{s\to\infty} \mu^{(2)}_{r,s}$), but the same does not hold for the variance of variance. Indeed, $\lim_{n\to\infty} \frac{\mathrm{M}^{(2)}_{h,n}}{n} > \lim_{s\to\infty} \mu^{(2)}_{h,s}$.

Of course, the returns and variance processes are not independent, and certain aspects of this dependence are reflected in a similar behaviour in their limiting distributions. In particular, recall that the forward and aggregated returns had identical regularity conditions and that whenever a moment of forward returns converges to a finite limit, the corresponding moment of aggregated returns converges to the normal value. Theorem 4 yields a similar result for forward and aggregated variances: the skewness of the aggregated variance converges to zero when the skewness of forward variance converges to a finite value. But for variance, the skewness regularity condition is slightly stronger: not only $\gamma \in (0,1)$ but also $c_4 \in (0,1)$ and $c_4 \neq \gamma$.

[10]Since proofs become increasingly lengthy we only state the limit of the conditional skewness of the aggregated conditional variance in the case that $\gamma \in (0,1)$, and for $\gamma \geq 1$ we present the principles of the derivation in Appendix A.4.

4 An Application: Approximate Predictive Distributions

Density forecasting is a prime application of our moment formulae. Whilst one must know all the moments of a random variable to determine its distribution,[11] it can be approximated based on the first few moments alone. Based on their relative merits and drawbacks, and given the frequency of their use in similar applications as well as the feasibility of obtaining an approximate distribution function in closed form, we have selected two approximation methods: the Edgeworth expansion and the Johnson SU distribution.

4.1 Distribution Approximations Methods

The Edgeworth expansion can approximate a density of interest around a base density, usually the standard normal density.[12] It belongs to the class of Gram-Charlier expansions, being a rearrangement of a Gram-Charlier A series.[13] However, Gram-Charlier A series and Edgeworth series are only equivalent asymptotically, when an infinite number of terms enter the expansions. In empirical applications using finite order approximations they can differ significantly, and the Edgeworth version is preferred since its error is controlled.[14] Nevertheless, the Edgeworth expansion may have monotonicity and convergence problems, i.e. the distribution function is not guaranteed to be monotonic and the error of approximation does not necessarily improve when we incerease the order of the expansion.

The first four terms of the Edgeworth expansion are:

$$f_x(x) \simeq f_x^E(x) = \varphi(x) - \frac{\tau_x}{6}\varphi^{(3)}(x) + \frac{(\kappa_x - 3)}{24}\varphi^{(4)}(x) + \frac{\tau_x^2}{72}\varphi^{(6)}(x),$$

where $f_x^E(x)$ is the second-order Edgeworth approximation of f_x, so moments (cumulants) of order higher than four (kurtosis) are ignored, φ is the standard normal density and $\varphi^{(k)}$ is its k^{th} derivative, and τ_x and κ_x denote the skewness and kurtosis of f_x. For our purposes f_x will be the density of the normalised forward returns.

A random variable x is said to follow a Johnson SU distribution if:

[11] To be precise, a distribution is uniquely determined by its moments only if the Carleman condition holds, i.e. only if $\sum_{n=1}^{\infty} \alpha_{2n}^{-0.5n} \to \infty$, where $\{\alpha_k\}$ is the moment sequence (see Serfling, 1980, p.46). In the following we assume that this condition is met.

[12] For the general theory and expansion see Edgeworth (1905), Wallace (1958) and Bhattacharya and Ghosh (1978).

[13] See Chebyshev (1860), Chebyshev (1890), Gram (1883), Charlier (1905) and Charlier (1906).

[14] The Edgeworth version of the expansion is actually a true asymptotic expansion, i.e. the error of approximation is controlled and it approaches zero as some parameter, e.g. the sample size for approximations of the sampling distribution of a random sample of size T, approaches infinity.

$$z = \gamma + \delta \sinh^{-1}\left(\frac{x-\xi}{\lambda}\right),$$

where z is a standard normal variable. The four parameters γ, δ, ξ and λ may be estimated using the moment-matching algorithm described in Tuenter (2001). Although flexible, the main disadvantage of this approach is that a Johnson SU distribution is not guaranteed to exist for any set of mean, variance, skewness and (positive) excess kurtosis.

4.2 Evaluation Methods

To assess how well these approximate distributions serve their purpose we should investigate whether they provide an adequate representation of the conditional distributions of forward returns. But these distributions are not observable, even ex-post, so we shall use simulated distributions as proxies. The null hypothesis is H_0: $F_m = F_s$, where F_m is the cumulative distribution function for the approximate distribution constructed using the first four moments and a specific approximation method, and F_s is the distribution function for the simulated forward returns based on the GARCH process. The simulated distribution F_s is given by the step-function of the sample. Thus, $F_s(x_i) = T^{-1}i$, where i = number of returns less than or equal to x_i; x_i with $i \in \{1, ..., T\}$ is an increasingly ordered sample, and T is the number of simulations.

Standard hypothesis tests where the null is the equality of two distributions are the Kolmogorov-Smirnov (KS) and the Anderson Darling (AD) tests. The KS test, proposed by Kolmogorov (1933), Smirnov (1939), Scheffe (1943) and Wolfowitz (1949) is, in effect, a simple hypothesis test which is based on the maximum difference between an empirical and a hypothetical cumulative distribution. The test statistic is given by KS $= \sqrt{T}D$, where D is the maximum distance between the two distributions, i.e. $D = \max_{1\le i\le T} |F_m(x_i) - F_s(x_i)|$. For practical implementations, a simpler variant of the statistic for the increasingly ordered sample is given by:

$$\text{KS} = \sqrt{T} \max_{1\le i\le T} \left\{ \max\left[F_m(x_i) - \tfrac{i-1}{T}, \tfrac{i}{T} - F_m(x_i) \right] \right\}.$$

When comparing alternative models, the one with the lowest KS value is deemed the most accurate for predicting the distribution in question.

The framework proposed by Anderson and Darling (1952) is more flexible, allowing for different weighting of the observations. They propose two distance measures, which are actually generalisations of the KS and Cramer von Mises (CVM) statistics.[15] The respective test statistics are given by:

$$\text{AD}_1 = T^{1/2} \max_{1 \leq i \leq T} \left| F_m(r_i) - F_s(r_i) \right| \left(\psi(F_m(r_i))^{1/2} \right), \tag{15}$$

$$\text{AD}_2 = T \sum_{i=1}^{T} \left[(F_m(r_i) - F_s(r_i))^2 \psi(F_m(r_i)) \right], \tag{16}$$

where ψ is a weighting function. Following convention we refer to the Anderson-Darling (AD) test as (16) with a weighting function $\psi(x) = (x(1-x))^{-1}$.[16]

Conducting these tests in our setting requires the simulation of critical values. The statistics only have standard distributions if the distribution under the null hypothesis is entirely pre-specified, but in our case the F_m distribution relies on estimated parameter values so the theoretical critical values are no longer applicable.

4.3 Data and Methodology

The performance of our proposed distribution forecasts is tested using daily observations on an equity index (S&P 500), a foreign exchange rate (Euro/dollar) and an interest rate (3-month Treasury bill). These series represent three major market risk types and within each class they represent the most important risk factors in terms of volumes of exposures. The three data sets used in this application were obtained from Datastream and each comprise 20 years of daily data from 1st January 1990 to 31st December 2009.[17] Figure 1 plots the daily log returns for the equity and exchange rate data and the daily changes in the interest rate.[18]

Table 1 presents the sample statistics of the empirical unconditional daily returns distribution over the entire sample. In accordance with stylized facts the mean of every series is not statistically different from zero and the unconditional volatility is highest for the equity and lowest for the interest rates. Skewness is negative and low (in absolute value) but significant for all three series, so extreme negative returns are more likely than extreme positive returns of the same magnitude, while excess

[15] The KS and Cramer-von Mises tests are obtained when $\psi(t) = 1$ in (15) and (16) respectively.

[16] For practical implementations and for an ordered sample, simpler variants of CVM and AD are given by:

$$\text{CVM} = \sum_{i=1}^{T} \left[F_m(x_i) - \tfrac{2i-1}{2T} \right]^2 + \tfrac{1}{12T}, \quad \text{AD} = -\sum_{i=1}^{T} \tfrac{2i-1}{T} \left[\ln(F_m(x_i)) + \ln(1 - F_m(x_{T+1-i})) \right] - T.$$

[17] The Euro was only introduced in 1999, so the ECU/dollar exchange rate is used before this date.

[18] First differences in fixed maturity interest rates are the equivalent of log returns on corresponding bonds.

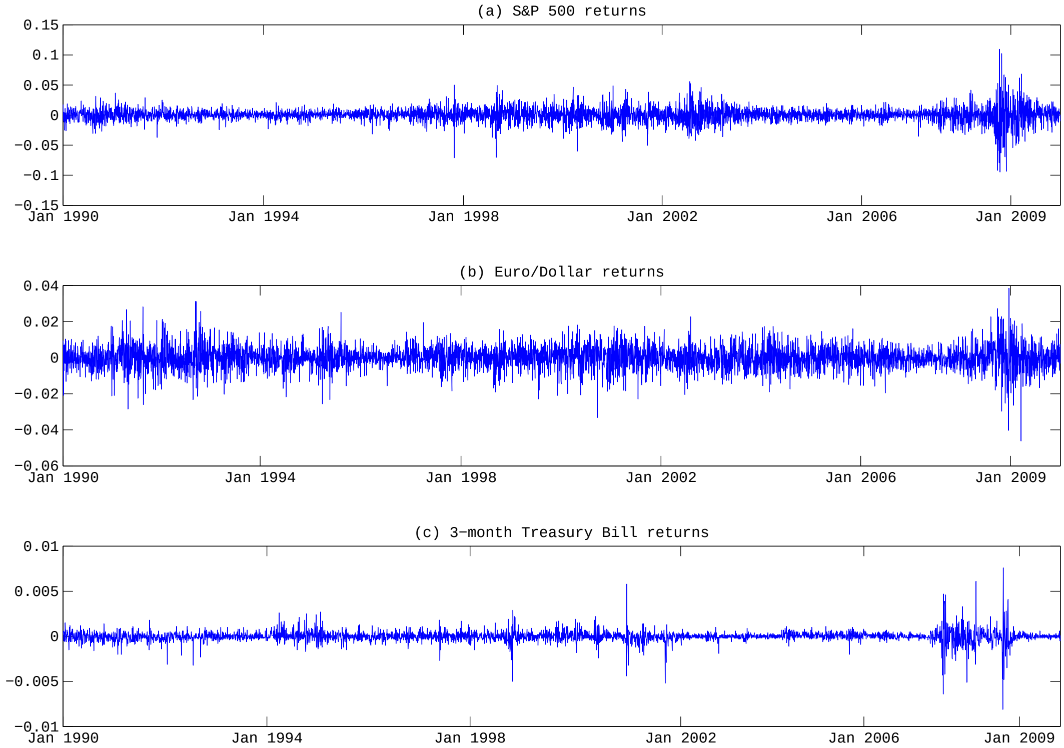

FIGURE 1: ***Returns*** *The equity and exchange rate daily (log) returns are computed as the first differences of the logarithm of the SP500 index values and Euro/dollar exchange rates, respectively. The interest rate returns are computed as first differences in interest rate values.*

kurtosis is always positive and highly significant, so the unconditional distributions of the series have a greater probability mass in the tails than the normal distribution with the same variance.

Four different GARCH models, namely the baseline GARCH(1,1) and the asymmetric GJR, each with normal and Student t error distributions, are estimated for each of the three time series. Their parameters are repeatedly estimated, approximately 2500 times, on a sample consisting of ten years of daily data which is rolled daily for an additional ten years. The resulting time series of model parameters are subsequently used to compute corresponding time series for the higher moments of forward and aggregated returns, based on the analytic formulae derived in Section 2. This way we may construct time series of conditional moments for the forward and aggregated returns and variances, for any time horizon h, from 3rd January 2000 to 31st December 2009.

For the symmetric models – the normal and Student t GARCH(1,1) – the skewness of the returns, both forward and aggregated, is zero by construction. However, the asymmetric specifications – the

	S&P 500	EUR/USD	3M IR
Mean	0.00022	-2E-05	-2E-05
Maximum	0.1	0.0384	0.0076
Minimum	-0.0947	-0.0462	-0.0081
Volatility	0.1862	0.1	0.0101
Skewness	-0.1981***	-0.0992**	-0.5007***
Excess kurtosis	9.1643***	2.7136***	28.2342***

TABLE 1: ***Summary statistics.****The summary statistics are of the equity and exchange rate daily log returns, and of the daily changes in interest rates from 1990 to 2009. Asterisks denote significance at 5% (*), 1% (**) and 0.1%(***). The standard error of the sample mean is equal to the sample standard deviation, divided by sample size. The standard errors under the null of normality are approximately* $(6/T)^{1/2}$ *and* $(24/T)^{1/2}$ *for the skewness and kurtosis, respectively, where T is the sample size. We used 252 risk days per year to annualize the standard deviation into volatility.*

normal and Student t GJR – lead to non-zero skewness forecasts for the aggregated returns. The skewness of the (forward and aggregated) variances is non-zero even for the symmetric models. All four models yield non-zero, positive excess kurtosis for all time series.

Now we apply the distribution approximation methods reviewed in Section 4.1 to derive distributions for the h-day forward returns. We combine four GARCH specifications (the normal and Student t GARCH(1,1) and GJR models) with two approximation methods (the Johnson SU distribution and the Edgeworth expansion) and thus have eight alternative approximate distributions to evaluate and compare with the simulated distributions, each based on 10,000 simulations. We test the accuracy of the approximations using the Kolmogorov-Smirnov (KS) distance, the Cramer von Mises (CVM) and Anderson Darling (AD) test statistics described in Section 4.2. To capture any differences between market regimes, the tests are performed for 150 days from a low volatility period (i.e. January to August 2006), 150 days from a high volatility period (i.e. August 2008 to March 2009) and the last 150 observations from 2009. In our results these periods are labelled 'low vol', 'high vol' and 'current' respectively. Finally, the time horizon we consider here is h = 5 days.

4.4 Empirical results

Table 2 summarizes the results of the distribution tests for each of the eight approximate distributions considered. The AD test gave results very similar to the CVM test, hence we only report the results for the KS and CVM tests. We report the mean values and the associated standard deviations of the test statistics and also the percentage of times when the computed test statistic was higher than the asymptotic 5% critical value. Since we perform the tests at the 5% significance level we expect a 5% rejection rate.

The 5% critical values are 0.0136 for the KS distance and 0.416 for the CVM statistic.[19] Although the asymptotic critical values do not apply exactly in our case, the model that produces the lowest values of the test statistic is still the best among the alternatives.[20] We now discuss the results in greater detail for the equity, exchange rate and interest rate distributions in turn.

(a) S&P 500 Index

The Johnson SU approximation appears to be a more suitable approximation than the Edgeworth expansion for the forward returns, especially when applied in combination with the moments produced by the Student t GARCH(1,1) and GJR models. For the normal GARCH(1,1) and GJR models the average values of both the KS and CVM test statistics are still greater, but only marginally, for the Edgeworth approximations, compared with their Johnson SU counterparts. Indeed, if the Edgeworth expansion is used, then the models with normal innovations fit the simulated distributions significantly better than their Student t counterparts. This is not the case for the Johnson SU distribution, where all four GARCH specifications provide very close fits to the simulated distributions (although again the normal models, and especially the normal GARCH(1,1), do give slightly better fits than the Student t models). On average, all models perform slightly better during the high volatility period than during the low volatility period. Overall, the Johnson SU normal GARCH(1,1) model produces the lowest average values of both the KS distance (0.0088) and CVM test statistic (0.1719).

[19]These are asymptotic results for a test where the distribution being tested for is continuous, fully known and generic (no particular family of distributions assumed). Stephens (1970) derives modified statistics for the finite sample case; however, with a sample size of 10,000 these modifications are not actually needed and the asymptotic results would apply, if the hypothetical distribution were fully specified. However, in our case this distribution is based on estimated results and we would need to simulate the correct critical values if we were to properly carry out the tests. Still, we report the percentage of times the test statistics are greater than the asymptotic critical values, so that we can infer, approximately, if the test results are at least in the vicinity of these asymptotic critical values. We also note that the results have to be interpreted with care since it is likely that the appropriate (simulated) critical values for this testing exercise are lower than the asymptotic critical values reported above (see Massey, 1951).

[20]What we mean by "best among alternatives" here means "closest to the (respective) simulated distribution". However, one has to interpret the results with care since the simulated distribution is obviously not the same for all alternative approximate distributions.

	Normal GARCH(1,1)				Student *t* GARCH(1,1)				Normal GJR				Student *t* GJR			
	total	low vol	high vol	current	total	low vol	high vol	current	total	low vol	high vol	current	total	low vol	high vol	current
S&P 500								Johnson SU								
avgKS	0.0088	0.0088	0.0086	0.0089	0.0092	0.0091	0.0091	0.0093	0.0088	0.0088	0.0087	0.0089	0.009	0.009	0.0089	0.0091
stdevKS	0.0027	0.0027	0.0026	0.0027	0.0027	0.0028	0.0026	0.0029	0.0027	0.0027	0.0025	0.0027	0.0027	0.0028	0.0025	0.0028
rejectKS	6.00%	6.00%	4.67%	7.33%	7.33%	8.67%	6.00%	7.33%	6.00%	6.67%	5.33%	6.00%	6.89%	8.00%	4.67%	8.00%
avgCVM	0.1719	0.1765	0.1577	0.1814	0.1897	0.1935	0.1809	0.1948	0.1737	0.1803	0.1593	0.1814	0.1824	0.1896	0.1688	0.1889
stdevCVM	0.1496	0.1628	0.1298	0.1543	0.1544	0.1664	0.1346	0.1611	0.1504	0.1635	0.1284	0.157	0.1528	0.1652	0.1296	0.1611
rejectCVM	6.44%	8.00%	4.67%	6.67%	6.00%	8.00%	4.00%	6.00%	6.89%	8.67%	4.67%	7.33%	6.00%	8.00%	4.00%	6.00%
								Edgeworth								
avgKS	0.0088	0.0088	0.0086	0.009	0.0171	0.01629	0.01676	0.0182	0.0089	0.0089	0.0088	0.009	0.0142	0.0142	0.0139	0.0144
stdevKS	0.0027	0.0027	0.0026	0.0027	0.0031	0.0029	0.0031	0.0031	0.0027	0.0027	0.0025	0.0028	0.003	0.0028	0.003	0.003
rejectKS	5.78%	6.00%	4.00%	7.33%	86.89%	80.67%	85.33%	94.67%	6.67%	8.00%	5.33%	6.67%	54.00%	54.67%	50.67%	56.67%
avgCVM	0.1721	0.1768	0.1579	0.1815	0.8535	0.7698	0.8121	0.9785	0.1764	0.1846	0.1614	0.1832	0.5387	0.5516	0.5059	0.5587
stdevCVM	0.1497	0.1629	0.1296	0.1546	0.3256	0.2887	0.2912	0.3561	0.1513	0.1647	0.1284	0.1582	0.2471	0.244	0.2278	0.2663
rejectCVM	6.22%	8.00%	4.67%	6.00%	91.33%	86.67%	91.33%	96.00%	6.67%	8.67%	4.67%	6.67%	56.89%	59.33%	50.00%	61.33%
Euro/dollar								Johnson SU								
avgKS	0.0088	0.0088	0.0086	0.009	0.009	0.009	0.0088	0.0091	0.0088	0.0088	0.0086	0.009	0.009	0.009	0.0088	0.0091
stdevKS	0.0027	0.0027	0.0026	0.0028	0.0027	0.0028	0.0025	0.0028	0.0027	0.0027	0.0026	0.0027	0.0027	0.0027	0.0025	0.0028
rejectKS	5.11%	4.67%	4.67%	6.00%	6.44%	8.00%	4.00%	7.33%	5.11%	4.67%	4.67%	6.00%	6.22%	8.00%	4.00%	6.67%
avgCVM	0.1713	0.1754	0.1571	0.1814	0.1802	0.1883	0.1636	0.1888	0.1713	0.1754	0.1571	0.1814	0.1803	0.1886	0.1646	0.1878
stdevCVM	0.1495	0.1618	0.1302	0.1546	0.1519	0.1647	0.129	0.1591	0.1496	0.1618	0.1303	0.1547	0.1519	0.1648	0.1297	0.1587
rejectCVM	6.44%	7.33%	4.67%	7.33%	6.89%	9.33%	4.67%	6.67%	6.44%	7.33%	4.67%	7.33%	5.55%	7.33%	3.33%	6.00%
								Edgeworth								
avgKS	0.0088	0.0088	0.0086	0.009	0.0136	0.0151	0.0122	0.0134	0.0088	0.0088	0.0086	0.009	0.0135	0.015	0.0122	0.0133
stdevKS	0.0027	0.0027	0.0026	0.0028	0.0032	0.0029	0.0029	0.003	0.0027	0.0027	0.0026	0.0027	0.0032	0.003	0.0029	0.0029
rejectKS	5.11%	4.67%	4.67%	6.00%	45.78%	67.33%	29.33%	40.67%	5.11%	4.67%	4.67%	6.00%	45.33%	66.67%	29.33%	40.00%
avgCVM	0.1713	0.1754	0.1571	0.1814	0.4941	0.6435	0.3761	0.4628	0.1713	0.1754	0.1571	0.1814	0.4886	0.644	0.3735	0.4484
stdevCVM	0.1495	0.1618	0.1302	0.1547	0.2628	0.2735	0.1876	0.2462	0.1496	0.1618	0.1303	0.1547	0.2621	0.2754	0.1878	0.2379
rejectCVM	6.44%	7.33%	4.67%	7.33%	52.89%	78.67%	32.00%	48.00%	6.44%	7.33%	4.67%	7.33%	44.89%	74.67%	24.00%	36.00%
3M Bill								Johnson SU								
avgKS	0.0106	0.0103	0.0104	0.0111	0.164	0.1315	0.2498	0.0989	0.0124	0.0113	0.0127	0.0132	0.0234	0.0212	0.0263	0.0227
stdevKS	0.0031	0.0031	0.0029	0.0032	0.0898	0.0238	0.0996	0.0214	0.0033	0.0032	0.003	0.0034	0.0047	0.0041	0.0046	0.0034
rejectKS	17.02%	16.67%	12.00%	23.58%	1	1	1	1	34.04%	26.00%	34.67%	43.09%	99.53%	98.67%	1	1
avgCVM	0.2546	0.2478	0.236	0.2856	155.3	85.868	312.3	48.57	0.3651	0.3038	0.3786	0.4234	1.782	1.4011	2.3533	1.5497
stdevCVM	0.1822	0.1865	0.1548	0.204	173.7	33.399	211.8	21.649	0.2154	0.2006	0.1921	0.2409	0.8393	0.6283	0.9624	0.435
rejectCVM	12.77%	13.33%	8.00%	17.89%	1	1	1	1	26.24%	19.33%	27.33%	33.33%	99.29%	98.00%	1	1
								Edgeworth								
avgKS	0.0267	0.0187	0.0325	0.0294	0.2662	0.2097	0.3491	0.234	0.0321	0.0217	0.0395	0.0356	0.1997	0.1496	0.254	0.1944
stdevKS	0.0074	0.0033	0.0054	0.004	0.0812	0.0247	0.0824	0.0181	0.0094	0.0035	0.0068	0.0044	0.0537	0.0179	0.0471	0.0122
rejectKS	98.35%	95.33%	1	1	1	1	1	1	99.76%	99.33%	1	1	1	1	1	1
avgCVM	2.6567	1.0831	3.9245	3.0298	286.1	181.8	438.1	227.9	4.0869	1.5392	6.1278	4.7048	188.6	99.35	295.4	167.1
stdevCVM	1.5653	0.4047	1.3952	0.8581	152.7	40.51	163.1	31	2.5031	0.5157	2.2873	1.2126	106.3	25.08	105.4	21.35
rejectCVM	99.29%	98.00%	1	1	1	1	1	1	1	1	1	1	1	1	1	1

TABLE 2: ***Distribution tests for the approximate distributions of 5-day forward returns***
Average KS distance and CVM test statistics, with asociatted standard deviations and the percentage of cases where the test statistics are greater than the asymptotic 5% CVs(reject) for the 5-day forward returns for the S&P 500, Euro/dollar, 3-month Treasury Bill

(b) Euro/Dollar Exchange Rate

From the results in Table 2 there is not much to choose between the two alternative approximation methods when the innovations are normal. Indeed, the results for both distribution tests are virtually identical (and good) for the normal GARCH(1,1) and GJR, using either the Johnson SU or Edgeworth expansion. For the Johnson SU approach, the fit is closest when the innovations are normal but the fit is almost as good based on the Student t GJR model. However, when the Edgeworth expansion is employed, the results obtained with the Student t models are significantly worse than when the innovations are normal.

(c) 3-month Treasury Bill Rate

As in (a), the Johnson SU should be preferred to the Edgeworth expansion. The Johnson SU approximation yields the lowest average KS distance as well as the lowest value for the CVM test statistic for the normal GARCH(1,1) model, the normal GJR model being second best. The fits deteriorate when innovations have a Student t distribution.

To summarize, the predictive distributions of forward returns on an equity index, an exchange rate and an interest rate may be well approximated using the analytic expressions for the first four moments that we have derived in this paper. The best distribution approximation method overall is the Johnson SU and all four GARCH models that we have tested have predictive forward returns distributions that can be well approximated, the easiest being that generated by the normal GARCH(1,1).

5 Conclusions

We have derived analytical expressions for the moments of forward and aggregated returns and variances for an established asymmetric GARCH specification, namely the GJR model, with a generic innovations distribution. Special cases include the normal and Student t GARCH(1,1) and GJR models. The distribution of forward returns is skewed only if the distribution of innovations is skewed, but the distribution of aggregated returns is skewed even if the innovation distribution is symmetric. The other source of skewness in this case is the asymmetric response of variance to positive and negative shocks (i.e. $\lambda \neq 0$). Furthermore, the conditional distribution of aggregated returns can exhibit both skewness and excess kurtosis even when the innovation distribution is normal.

There are two sources of kurtosis in forward returns: the degree of leptokurtosis of the innovation distribution and the uncertainty in forward variance. Since the one-step-ahead variance is deterministic in a GARCH setting, the kurtosis coefficient of the one-step-ahead returns distributions is always equal to that of the innovation distribution. However, whenever we forecast s-steps ahead (with $s > 1$) using a GARCH(1,1) or GJR model, the s-step-ahead returns distribution for $s > 1$ will always have a higher kurtosis than the one-step-ahead returns distribution, due to the positive conditional variance of the conditional variance. Uncertainty in variance increases the probability mass in the tails of the forward one-period returns distribution, and time-varying uncertainty in variance (time-varying conditional variance of the conditional variance) introduces dynamics in the higher moments of the forward returns.

Provided the unconditional moments exist (i) the conditional moments of forward returns converge to the corresponding unconditional moments as the time horizon increases, and (ii) the conditional moments of aggregated returns converge to the corresponding moments of a normal distribution. Otherwise, the moments of both the forward returns as well as the aggregated returns generally diverge to (plus or minus) infinity.

An empirical application computes higher moments of the forward and aggregated returns and variances of the S&P 500 index, the Euro/dollar exchange rate and the 3-month US Treasury bill rate, using our analytic expressions for the first four moments based on four different GARCH processes. Subsequently, we approximate predictive distributions for forward returns using these higher moment forecasts and the Johnson SU distribution or the Edgeworth expansion. A variety of statistical tests evaluate the accuracy of these approximations, relative to the corresponding simulated GARCH returns distributions. The results of these tests are in general very good for the vast majority of the approximate distributions of the forward returns. Hence, our moment expressions may have useful applications to financial problems which, until now, have required GARCH returns distributions to be simulated.

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APPENDICES: CONDITIONAL MOMENTS FOR THE GENERIC GJR MODEL

The model's s-step ahead conditional variances are given by:

$$h_{t+s} = \omega + \left(\alpha + \lambda I_{t+s-1}^{-}\right) \varepsilon_{t+s-1}^{2} + \beta h_{t+s-1} \tag{17}$$

The aim of this appendix is to calculate the conditional moments of the return the forward rreturn r_{t+s} and of its conditional variance h_{t+s}, as well as of the aggregated return and of its conditional variance according to the model. Specifically, for i = 1, 2, 3 and 4, and for $x = r$ and h, we compute:

$$\tilde{\mu}_{x,s}^{(i)} = E_t\left(x_{t+s}^{i}\right), \quad \mu_{x,s}^{(i)} = E_t\left(\left(x_{t+s} - \tilde{\mu}_{x,s}^{(1)}\right)^{i}\right),$$

$$\tilde{M}_{x,n}^{(i)} = E_t\left[\left(\sum_{s=1}^{n} x_{t+s}\right)^{i}\right], \quad M_{x,n}^{(i)} = E_t\left(\left(\sum_{s=1}^{n}\left(x_{t+s} - \tilde{\mu}_{x,s}^{(1)}\right)\right)^{i}\right),$$

and the corresponding standardized moments:

$$\tau_{x,s} = \frac{\mu_{x,s}^{(3)}}{\left(\mu_{x,s}^{(2)}\right)^{3/2}}, \quad \mathrm{T}_{x,n} = \frac{M_{x,n}^{(3)}}{\left(M_{x,n}^{(2)}\right)^{3/2}}, \quad \kappa_{x,s} = \frac{\mu_{x,s}^{(4)}}{\left(\mu_{x,s}^{(2)}\right)^{2}}, \quad \mathrm{K}_{x,n} = \frac{M_{x,n}^{(4)}}{\left(M_{x,n}^{(2)}\right)^{2}}.$$

We focus on calculating the un-centred conditional moments; the centred moments will follow using simple formulae. We denote by F_0 the distribution function for $D\,(0,1)$, evaluated at zero, and set:

$$\varphi = \alpha + \lambda F_0 + \beta \tag{18}$$

and $\bar{h} = \omega(1-\varphi)^{-1}$. For both the normal and the standardized Student *t*, $F_0 = \frac{1}{2}$, since the two distributions are symmetric, thus for these two special cases φ becomes:

$$\varphi = \alpha + \frac{\lambda}{2} + \beta \tag{19}$$

The following notation will also be useful:

$$\tilde{\mu}_{h,su}^{(i,j)} = E_t\left(h_{t+s}^{i} h_{t+s+u}^{j}\right)$$

$$\tilde{\mu}_{h,suv}^{(i,j,k)} = E_t\left(h_{t+s}^{i} h_{t+s+u}^{j} h_{t+s+u+v}^{k}\right)$$

$$\tilde{\mu}_{h,suvw} = E_t\left(h_{t+s} h_{t+s+u} h_{t+s+u+v} h_{t+s+u+v+w}\right)$$

$$\theta_{su}^{(j)} = E_t\left(\varepsilon_{t+s} h_{t+s+u}^{j}\right)$$

where $s, u, v, w > 0$.

After deriving the formulae for the generic model, we allow the innovations distribution $D(0,1)$ to take two specific functional forms, largely used in practice: the standard normal and the (standardized) Student *t*.[21] When $D(0,1)$ is the standard normal distribution the moments with odd order are all zero and the even moments are given by $\mu_z^{(2r)} = \prod_{i=1}^{r}(2i-1)$. When $D(0,1)$ is the standardized Student *t* distribution, the odd order moments are again all zero (as long as the number of degrees of freedom $\nu > r$, the order of the moment) and the even moments are given by: $\mu_z^{(2r)} = (\nu-2)^r \frac{\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\nu-r\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\nu\right)}$.

A.1: Conditional Moments of Forward and Aggregated Returns

(a) First Moments

For $s \geq 1$ and using the notation $R_{tn} = \sum_{s=1}^{n} r_{t+s}$ for the aggregated return and also using the tower law of expectations, we get:

$$E_t(r_{t+s}) = E_t(\mu + \varepsilon_{t+s}) = \mu + E_t\left(\underbrace{E_{t+s-1}(\varepsilon_{t+s})}_{0}\right) = \mu$$

$$E_t(R_{tn}) = E_t\left(\sum_{s=1}^{n} r_{t+s}\right) = \sum_{s=1}^{n} E_t(r_{t+s}) = n\mu$$

(b) Second Moments

The second moment of the forward return is:

$$E_t\left(r_{t+s}^2\right) = E_t\left[(\mu+\varepsilon_{t+s})^2\right] = \mu^2 + \bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right),$$

where we used the expression for the second moment of variance $\tilde{\mu}_{h,s}^{(1)} = \mu_{r,s}^{(2)} = \bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right)$ derived in Appendix A.2 below. The second moment of the aggregated return is:

$$E_t\left(R_{tn}^2\right) = E_t\left(\sum_{s=1}^{n} r_{t+s}\right)^2 = E_t\left(\sum_{s=1}^{n} r_{t+s}^2 + 2\sum_{s=1}^{n}\sum_{u=1}^{n-s} r_{t+s}r_{t+s+u}\right).$$

$$E_t\left(\sum_{s=1}^{n} r_{t+s}^2\right) = \sum_{s=1}^{n}\left(\mu^2 + \bar{h} + \varphi^{s-1}\left(h_{t+1}-\bar{h}\right)\right) = n\left(\mu^2+\bar{h}\right) + \left(h_{t+1}-\bar{h}\right)(1-\varphi)^{-1}(1-\varphi^n)$$

$$E_t\left(\sum_{s=1}^{n}\sum_{u=1}^{n-s} r_{t+s}r_{t+s+u}\right) = \sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t(r_{t+s}r_{t+s+u}) = \sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left((\mu+\varepsilon_{t+s})(\mu+\varepsilon_{t+s+u})\right) = {}^1\!/_2\, n(n-1)\mu^2$$

Hence, the expression for the second moment of aggregated returns becomes:

$$E_t\left(R_{tn}^2\right) = n^2\mu^2 + n\bar{h} + \left(h_{t+1}-\bar{h}\right)(1-\varphi)^{-1}(1-\varphi^n)$$

[21] We shall only derive the expressions for the two special cases when a particular expression derived for the generic case differs for one (or both) of the special cases; when no formulae for the special cases are mentioned, the generic formulae apply.

(c) Third Moments

For the third moment of forward returns we write:

$$E_t\left(r_{t+s}^3\right) = E_t\left[\left(\mu+\varepsilon_{t+s}\right)^3\right] = E_t\left(\mu^3+3\mu^2\varepsilon_{t+s}+3\mu\varepsilon_{t+s}^2+\varepsilon_{t+s}^3\right)$$
$$= \mu^3+3\mu\tilde{\mu}_{h,s}^{(1)}+E_t\left(z_{t+s}^3 h_{t+s}^{3/2}\right) = \mu^3+3\mu\tilde{\mu}_{h,s}^{(1)}+\tau_z E_t\left(h_{t+s}^{3/2}\right).$$

For both the normal and Student *t* GJR (or rather for any GJR model with a symmetric innovations distribution), the expression for the third moment of returns is given by:

$$E_t\left(r_{t+s}^3\right) = \mu^3+3\mu\tilde{\mu}_{h,s}^{(1)} = \mu^3+3\mu\left(\bar{h}+\varphi^{s-1}\left(h_{t+1}-\bar{h}\right)\right)$$

However, for the generic, skewed model, we need to approximate $E_t\left(h_{t+s}^{3/2}\right)$ using a second order Taylor series expansion. In general, for a smooth function $g(X)$:

$$g(X) \approx g\left(E_t(X)\right)+g'\left(E_t(X)\right)\left(X-E_t(X)\right)+{}^1\!/_2 g''\left(E_t(X)\right)\left(X-E_t(X)\right)^2.$$

Taking expectations we get: $E_t\left(g(X)\right) \approx g\left(E_t(X)\right)+{}^1\!/_2 g''\left(E_t(X)\right)V_t(X)$. Setting $g(X)=X^{3/2}$, so $g'(X)=\frac{3}{2}X^{1/2}$ and $g''(X)=\frac{3}{4}X^{-1/2}$ and setting $X=h_{t+s}$ yields:

$$E_t\left(h_{t+s}^{3/2}\right) \simeq \tfrac{5}{8}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3/2}+\tfrac{3}{8}\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-1/2},$$

where the expressions for $\tilde{\mu}_{h,s}^{(1)}$ and $\tilde{\mu}_{h,s}^{(2)}$ are given in Appendix A.2 below.

We now compute the third moment of the aggregated returns:

$$E_t\left(R_{tn}^3\right) = E_t\left(\sum_{s=1}^{n} r_{t+s}\right)^3 = \sum_{s=1}^{n} E_t\left(r_{t+s}^3\right)+3\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left[E_t\left(r_{t+s}^2 r_{t+s+u}\right)+E_t\left(r_{t+s}r_{t+s+u}^2\right)\right]$$
$$+6\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u} E_t\left(r_{t+s}r_{t+s+u}r_{t+s+u+v}\right)$$

$$\sum_{s=1}^{n} E_t\left(r_{t+s}^3\right) = n\mu\left(\mu^2+3\bar{h}\right)+3\mu(1-\varphi)^{-1}\left(1-\varphi^n\right)\left(h_{t+1}-\bar{h}\right)+\tau_z\sum_{s=1}^{n} E_t\left(h_{t+s}^{3/2}\right)$$

$$\sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(r_{t+s}^2 r_{t+s+u}\right) = \sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left[\left(\mu^2+2\mu\varepsilon_{t+s}+\varepsilon_{t+s}^2\right)\left(\mu+\varepsilon_{t+s+u}\right)\right] = \sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\mu^3+\mu\mu_{h,s}^{(1)}\right)$$
$$= \mu\left[\tfrac{n(n-1)}{2}\left(\mu^2+\bar{h}\right)+(1-\varphi)^{-1}\left[n-(1-\varphi)^{-1}\left(1-\varphi^n\right)\right]\left(h_{t+1}-\bar{h}\right)\right]$$

$$\sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(r_{t+s}r_{t+s+u}^2\right) = \sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(\left(\mu+\varepsilon_{t+s}\right)\left(\mu^2+2\mu\varepsilon_{t+s+u}+\varepsilon_{t+s+u}^2\right)\right)$$
$$= \sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\mu^3+\mu\tilde{\mu}_{h,s+u}^{(1)}+E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right)\right)$$

$$E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right) = E_t\left(\varepsilon_{t+s}E_{t+s+u-1}\left(\varepsilon_{t+s+u}^2\right)\right) = \theta_{su}^{(1)}$$
$$= E_t\left(\varepsilon_{t+s}\left(\omega+\left(\alpha+\lambda I_{t+s+u-1}\right)\varepsilon_{t+s+u-1}^2+\beta h_{t+s+u-1}\right)\right) = \varphi E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u-1}^2\right)$$
$$= \varphi^{u-1}E_t\left(\varepsilon_{t+s}h_{t+s+1}\right) = \varphi^{u-1}\left(\alpha\tau_z+\lambda\int_{z=-\infty}^{0} z^3 f(z)\,dz\right)E_t\left(h_{t+s}^{3/2}\right)$$

where f is the *pdf* of the innovation distribution; the final expression for $\sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(r_{t+s}r_{t+s+u}^2\right)$ becomes:

$$\sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(r_{t+s}r_{t+s+u}^2\right)=\tfrac{n(n-1)}{2}\mu\left(\mu^2+\bar{h}\right)+(1-\varphi)^{-1}\left(\begin{array}{l}\mu\left[\varphi(1-\varphi)^{-1}\left(1-\varphi^n\right)-n\varphi^n\right]\left(h_{t+1}-\bar{h}\right)\\+\left(\alpha\tau_z+\lambda\int\limits_{z=-\infty}^{0}z^3f\left(z\right)dz\right)\sum\limits_{s=1}^{n}(1-\varphi^{n-s})E_t\left(h_{t+s}^{3/2}\right)\end{array}\right)$$

For the normal GJR we have $\tau_z=0$ and also it can be easily shown that:

$$\int\limits_{z=-\infty}^{0}z^3f\left(z\right)dz=\int\limits_{z=-\infty}^{0}\tfrac{1}{\sqrt{2\pi}}z^3\exp\left(-\tfrac{z^2}{2}\right)dz=-\sqrt{\tfrac{2}{\pi}}.$$

Similarly, for the Student *t* GJR, we have $\tau_z=0$ and easily get:

$$\int\limits_{z=-\infty}^{0}z^3f\left(z\right)=\int\limits_{z=-\infty}^{0}z^3\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi(\nu-2)}}\left(1+\tfrac{z^2}{\nu-2}\right)^{-\frac{\nu+1}{2}}dz=-\tfrac{2}{\sqrt{\pi}}\tfrac{(\nu-2)^{3/2}}{(\nu-1)(\nu-3)}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)},$$

Repeatedly applying the tower law, we get that:

$$\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}E_t\left(r_{t+s}r_{t+s+u}r_{t+s+u+v}\right)=\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\mu^3=\tfrac{n(n-1)(n-2)}{6}\mu^3.$$

(d) Fourth Moments

The fourth moment of the forward returns is

$$E_t\left(r_{t+s}^4\right)=E_t\left(\mu^4+4\mu^3\varepsilon_{t+s}+6\mu^2\varepsilon_{t+s}^2+4\mu\varepsilon_{t+s}^3+\varepsilon_{t+s}^4\right)=\mu^4+6\mu^2\tilde{\mu}_{h,s}^{(1)}+4\mu\tau_zE_t\left(h_{t+s}^{3/2}\right)+\kappa_z\tilde{\mu}_{h,s}^{(2)},$$

where $\tilde{\mu}_{h,s}^{(1)}$ and $\tilde{\mu}_{h,s}^{(2)}$ are derived in Appendix A.2 below and $E_t\left(h_{t+s}^{3/2}\right)$ is given above as a function of these first two conditional moments of the forward variance.

In the special case that the innovation distribution is the standard normal, $E_t\left(r_{t+s}^4\right)=\mu^4+6\mu^2\tilde{\mu}_{h,s}^{(1)}+3\tilde{\mu}_{h,s}^{(2)}$, while for the Student *t* GJR, $E_t\left(r_{t+s}^4\right)=\mu^4+6\mu^2\tilde{\mu}_{h,s}^{(1)}+3\tfrac{\nu-2}{\nu-4}\tilde{\mu}_{h,s}^{(2)}$.

For the fourth moment of aggregated returns, we write:

$$\begin{aligned}E_t\left(R_{tn}^4\right)=&\sum_{s=1}^{n}E_t\left(r_{t+s}^4\right)+\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left[4\left(E_t\left(r_{t+s}^3r_{t+s+u}\right)+E_t\left(r_{t+s}r_{t+s+u}^3\right)\right)+6E_t\left(r_{t+s}^2r_{t+s+u}^2\right)\right]\\&+12\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\left(E_t\left(r_{t+s}^2r_{t+s+u}r_{t+s+u+v}\right)+E_t\left(r_{t+s}r_{t+s+u}^2r_{t+s+u+v}\right)+E_t\left(r_{t+s}r_{t+s+u}r_{t+s+u+v}^3\right)\right)\\&+24\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\sum_{w=1}^{n-s-u-v}E_t\left(r_{t+s}r_{t+s+u}r_{t+s+u+v}r_{t+s+u+v+w}\right)\end{aligned}$$

$$\begin{aligned}\sum_{s=1}^{n}E_t\left(r_{t+s}^4\right)&=\sum_{s=1}^{n}\left(\mu^4+6\mu^2\tilde{\mu}_{h,s}^{(1)}+4\mu\tau_zE_t\left(h_{t+s}^{3/2}\right)+\kappa_z\tilde{\mu}_{h,s}^{(2)}\right)\\&=n\mu^2\left(\mu^2+6\bar{h}\right)+6\mu^2(1-\varphi)^{-1}\left(1-\varphi^n\right)\left(h_{t+1}-\bar{h}\right)+\sum_{s=1}^{n}\left(4\mu\tau_zE_t\left(h_{t+s}^{3/2}\right)+\kappa_z\tilde{\mu}_{h,s}^{(2)}\right)\end{aligned}$$

For the normal GJR, $\sum_{s=1}^{n}E_t\left(r_{t+s}^4\right)=n\mu^2\left(\mu^2+6\bar{h}\right)+6\mu^2(1-\varphi)^{-1}\left(1-\varphi^n\right)\left(h_{t+1}-\bar{h}\right)+3\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)}$,

while for the Student *t* GJR the sum above becomes:

$$\sum_{s=1}^{n}E_t\left(r_{t+s}^4\right)=n\mu^2\left(\mu^2+6\bar{h}\right)+6\mu^2(1-\varphi)^{-1}\left(1-\varphi^n\right)\left(h_{t+1}-\bar{h}\right)+3\tfrac{\nu-2}{\nu-4}\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)}.$$

$$\sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(r_{t+s}^3 r_{t+s+u}\right)=\sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(\left(\mu^3+3\left(\mu^2\varepsilon_{t+s}+\mu\varepsilon_{t+s}^2\right)+\varepsilon_{t+s}^3\right)\left(\mu+\varepsilon_{t+s+u}\right)\right)$$
$$=\tfrac{n(n-1)}{2}\mu^2\left(\mu^2+3\bar{h}\right)+3\mu^2(1-\varphi)^{-1}\left[n-(1-\varphi)^{-1}\left(1-\varphi^n\right)\right]\left(h_{t+1}-\bar{h}\right)$$
$$+\mu\tau_z\sum_{s=1}^{n}(n-s)\,E_t\left(h_{t+s}^{3/2}\right),$$

which for the normal and Student *t* GJR models becomes:

$$\sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(r_{t+s}^3 r_{t+s+u}\right)=\tfrac{n(n-1)}{2}\mu^2\left(\mu^2+3\bar{h}\right)+3\mu^2(1-\varphi)^{-1}\left[n-(1-\varphi)^{-1}\left(1-\varphi^n\right)\right]\left(h_{t+1}-\bar{h}\right).$$

$$\sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(r_{t+s}r_{t+s+u}^3\right)=\sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(\left(\mu+\varepsilon_{t+s}\right)\left(\mu^3+3\left(\mu^2\varepsilon_{t+s+u}+\mu\varepsilon_{t+s+u}^2\right)+\varepsilon_{t+s+u}^3\right)\right)$$
$$=\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\mu^4+3\,\mu^2\tilde{\mu}_{h,s+u}^{(1)}+\mu\tau_z E_t\left(h_{t+s+u}^{3/2}\right)+3\mu E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right)+E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^3\right)\right)$$

Now, $E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^3\right)=E_t\left(\varepsilon_{t+s}E_{t+s+u-1}\left(z_{t+s+u}^3 h_{t+s+u}^{3/2}\right)\right)=\tau_z E_t\left(\varepsilon_{t+s}h_{t+s+u}^{3/2}\right)=\tau_z\theta_{su}^{(3/2)}$, which for the normal and Student *t* GJR models reduces to: $E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^3\right)=0$, since $\tau_z=0$.[22] To solve for $\theta_{su}^{(3/2)}$, we are using a second order Taylor expansion around $\tilde{\mu}_{h,s+u}^{(1)}$, and obtain:

$$h_{t+s+u}^{3/2}\simeq\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{3/2}+\tfrac{3}{2}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{1/2}\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right)+\tfrac{3}{8}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right)^2,$$

which yields:

$$E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^3\right)=\tau_z\theta_{su}^{(3/2)}=\tfrac{3}{4}\tau_z\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{1/2}\left(E_t\left(\varepsilon_{t+s}h_{t+s+u}\right)+\tfrac{1}{2}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1}E_t\left(\varepsilon_{t+s}h_{t+s+u}^2\right)\right),$$
$$\theta_{su}^{(2)}=E_t\left(\varepsilon_{t+s}\left(\omega+\left(\alpha+\lambda I_{t+s+u-1}^{-}\right)\varepsilon_{t+s+u-1}^2+\beta h_{t+s+u-1}\right)^2\right)=\gamma\theta_{s(u-1)}^{(2)}+2\omega\varphi\theta_{s(u-1)}^{(1)},$$

where we have used that, conditional on the information available at time t, the indicator function I_t^- is independent of all (contemporaneous) ε_t^{2k} for any natural number k. We also set:[23]

$$\gamma=\left(\alpha^2+\left(2\alpha\lambda+\lambda^2\right)F_0\right)\kappa_z+\beta^2+2\beta\left(\alpha+\lambda F_0\right)=\varphi^2+\left(\kappa_z-1\right)\left(\alpha+\lambda F_0\right)^2+\kappa_z\lambda^2F_0\left(1-F_0\right)$$

If $D(0,1)$is the standard normal distribution, γ becomes:

$$\gamma=\varphi^2+2\left(\alpha+\tfrac{\lambda}{2}\right)^2+\tfrac{3}{4}\lambda^2.$$

When $D(0,1)$ is the standardized Student *t* distribution, γ is given by:

$$\gamma=\varphi^2+\left(3\tfrac{\nu-2}{\nu-4}-1\right)\left(\alpha+\tfrac{\lambda}{2}\right)^2+\tfrac{3}{4}\left(\tfrac{\nu-2}{\nu-4}\right)\lambda^2$$

Solving the recursion for $\theta_{su}^{(2)}$, we get:

$$\theta_{su}^{(2)}=\gamma^{u-1}\theta_{s1}^{(2)}+2\omega\varphi\sum_{j=1}^{u-1}\gamma^{j-1}\theta_{s(u-j)}^{(1)}.$$

[22]Even though for the normal and Student *t* GJR $\tau_z=0$, $\theta_{su}^{(3/2)}=E_t\left(\varepsilon_{t+s}h_{t+s+u}^{3/2}\right)$ is generally non-zero for these models (and enters the expressions of higher moments computed below) and this is why we still consider the normal and Student *t* special cases in the derivation of $\theta_{su}^{(3/2)}$.

[23]It can be shown, using the Cauchy – Buniakowsky – Schwarz inequality, that the kurtosis is always greater than or equal to 1, hence $\kappa_z\geq 1$. Now it can be easily seen that $\gamma>0$.

$\sum_{j=1}^{u-1}\gamma^{j-1}\theta_{s(u-j)}^{(1)} = c_9\sum_{j=1}^{u-1}\gamma^{j-1}\varphi^{u-j-1}E_t\left(h_{t+s}^{3/2}\right) = c_9(\varphi-\gamma)^{-1}\left(\varphi^{u-1}-\gamma^{u-1}\right)E_t\left(h_{t+s}^{3/2}\right)$, where
$c_9 = \left(\alpha\tau_z + \lambda\int_{x=-\infty}^{0}x^3 f(x)\,dx\right)$.

$$\begin{aligned}\theta_{s1}^{(2)} &= E_t\left(\varepsilon_{t+s}h_{t+s+1}^2\right) = E_t\left(\varepsilon_{t+s}\left(\omega + \left(\alpha+\lambda I_{t+s}^-\right)\varepsilon_{t+s}^2 + \beta h_{t+s}\right)^2\right)\\ &= \left(\alpha\left(\alpha\mu_z^{(5)} + 2\beta\tau_z\right) + \lambda(2\alpha+\lambda)\int_{x=-\infty}^{0}x^5 f(x)\,dx + 2\lambda\beta\int_{x=-\infty}^{0}x^3 f(x)\,dx\right)E_t\left(h_{t+s}^{5/2}\right)\\ &\quad +2\left(\omega\alpha\tau_z + \lambda\omega\int_{x=-\infty}^{0}x^3 f(x)\,dx\right)E_t\left(h_{t+s}^{3/2}\right)\end{aligned}$$

For the normal GJR, $\tau_z = \mu_z^{(5)} = 0$, $f(z) = \varphi(z)$, and it can be easily shown that $\int_{z=-\infty}^{0} z^5\varphi(z)\,dz = -4\sqrt{\frac{2}{\pi}}$. Similarly, for the Student *t* GJR, we again have have $\tau_z = \mu_z^{(5)} = 0$ and $f(z) = f_\nu(z)$. After some algebraic manipulation, we get that: $\int_{z=-\infty}^{0} z^5 f_v(z) = -\frac{8}{\sqrt{\pi}}\frac{(\nu-2)^{5/2}}{(\nu-1)(\nu-3)(\nu-5)}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$.

Thus, the final expression for $\theta_{su}^{(3/2)}$ becomes:

$$\theta_{su}^{(3/2)} = \frac{3}{4}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{1/2}\left(\begin{array}{l} c_9\varphi^{u-1}E_t\left(h_{t+s}^{3/2}\right) + \frac{1}{2}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1}\\ \left(c_{10}\gamma^{u-1}E_t\left(h_{t+s}^{5/2}\right) + 2\omega c_9\left(\varphi(\varphi-\gamma)^{-1}\left(\varphi^{u-1}-\gamma^{u-1}\right)+\gamma^{u-1}\right)E_t\left(h_{t+s}^{3/2}\right)\right)\end{array}\right),$$

with $c_{10} = \alpha\left(\alpha\mu_z^{(5)} + 2\beta\tau_z\right) + \lambda(2\alpha+\lambda)\int_{x=-\infty}^{0}x^5 f(x)\,dx + 2\beta\lambda\int_{x=-\infty}^{0}x^3 f(x)\,dx$ and $E_t\left(h_{t+s}^{5/2}\right)$ is given approximately using a second order Taylor expansion for $h_{t+s}^{5/2}$ around $E_t(h_{t+s})$:
$E_t\left(h_{t+s}^{5/2}\right) \simeq \frac{1}{8}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{1/2}\left(15\tilde{\mu}_{h,s}^{(2)} - 7\left(\tilde{\mu}_{h,s}^{(1)}\right)^2\right)$.

$$\begin{aligned}\sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(r_{t+s}^2 r_{t+s+u}^2\right) &= \sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left[(\mu+\varepsilon_{t+s})^2(\mu+\varepsilon_{t+s+u})^2\right]\\ &= \sum_{s=1}^{n}\sum_{u=1}^{n-s}\left[\mu^4 + \mu^2\left(\tilde{\mu}_{h,s}^{(1)} + \tilde{\mu}_{h,s+u}^{(1)}\right) + 2\mu E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right) + E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right)\right],\end{aligned}$$

$$\begin{aligned}E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right) &= E_t\left(\varepsilon_{t+s}^2 h_{t+s+u}\right) = \omega\tilde{\mu}_{h,s}^{(1)} + \varphi E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u-1}^2\right)\\ &= \omega\tilde{\mu}_{h,s}^{(1)}\left((1-\varphi)^{-1}(1-\varphi^u) - \varphi^{u-1}\right) + \varphi^{u-1}E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+1}^2\right), \text{ with}\end{aligned}$$

$$\begin{aligned}E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+1}^2\right) &= E_t\left(\varepsilon_{t+s}^2 h_{t+s+1}\right) = E_t\left(\varepsilon_{t+s}^2\left(\omega + \left(\alpha+\lambda I_{t+s}^-\right)\varepsilon_{t+s}^2 + \beta h_{t+s}\right)\right)\\ &= \omega\tilde{\mu}_{h,s}^{(1)} + \kappa_z\left(\alpha + \lambda F_0 + \kappa_z^{-1}\beta\right)\tilde{\mu}_{h,s}^{(2)}\end{aligned}$$

The final expression for $E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right)$ becomes:

$E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right) = \bar{h}(1-\varphi^u)\tilde{\mu}_{h,s}^{(1)} + \varphi^{u-1}\kappa_z\left(\alpha + \lambda F_0 + +\kappa_z^{-1}\beta\right)\tilde{\mu}_{h,s}^{(2)}$.

The expressions for the normal and Student *t* GJR are obtained by replacing $\kappa_z = 3$ and $\kappa_z = 3\frac{(\nu-2)}{(\nu-4)}$, respectively, and $F_0 = \frac{1}{2}$ in the expression above.

$$E_t\left(r_{t+s}^2 r_{t+s+u} r_{t+s+u+v}\right) = E_t\left(r_{t+s}^2 r_{t+s+u} E_{t+s+u+v-1}\left(r_{t+s+u+v}\right)\right) = \mu E_t\left(r_{t+s}^2 r_{t+s+u}\right) = \mu^2 E_t\left(r_{t+s}^2\right) = \mu^2\left(\mu^2 + \tilde{\mu}_{h,s}^{(1)}\right)$$

$$E_t\left(r_{t+s} r_{t+s+u}^2 r_{t+s+u+v}\right) = \mu E_t\left(\left(\mu + \varepsilon_{t+s}\right)\left(\mu^2 + h_{t+s+u}\right)\right) = \mu^4 + \mu^2 \tilde{\mu}_{h,s}^{(1)} + \mu \theta_{su}^{(1)}$$

$$\begin{aligned} E_t\left(r_{t+s} r_{t+s+u} r_{t+s+u+v}^2\right) &= E_t\left(r_{t+s} r_{t+s+u} E_{t+s+u+v-1}\left(\mu^2 + 2\mu\varepsilon_{t+s+u+v} + \varepsilon_{t+s+u+v}^2\right)\right) \\ &= E_t\left(r_{t+s} r_{t+s+u}\left(\mu^2 + h_{t+s+u+v}\right)\right) \\ &= \mu^4 + \mu^2 \tilde{\mu}_{h,(s+u+v)}^{(1)} + \mu \theta_{s(u+v)}^{(1)} + \mu E_t\left(\varepsilon_{t+s+u} h_{t+s+u+v}\right) + E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u} h_{t+s+u+v}\right) \end{aligned}$$

$E_t\left(\varepsilon_{t+s+u} h_{t+s+u+v}\right) = E_t\left(E_{t+s}\left(\varepsilon_{t+s+u} h_{t+s+u+v}\right)\right) = E_t\left(E_{t_1}\left(\varepsilon_{t_1+u} h_{t_1+u+v}\right)\right)$, where $t_1 = t+s$.

We showed that $E_t\left(\varepsilon_{t+s} h_{t+s+u}\right) = \theta_{su}^{(1)} = c_9 \varphi^{u-1} E_t\left(h_{t+s}^{3/2}\right)$. Hence

$E_{t_1}\left(\varepsilon_{t_1+u} h_{t_1+u+v}\right) = c_9 \varphi^{v-1} E_{t_1}\left(h_{t_1+u}^{3/2}\right)$ and thus we get:

$E_t\left(\varepsilon_{t+s+u} h_{t+s+u+v}\right) = c_9 \varphi^{v-1} E_t\left(E_{t_1}\left(h_{t_1+u}^{3/2}\right)\right) = c_9 \varphi^{v-1} E_t\left(h_{t+s+u}^{3/2}\right)$.

$$\begin{aligned} E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}^2\right) &= E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u} h_{t+s+u+v}\right) = E_t\left(\varepsilon_{t+s} E_{t_1}\left(\varepsilon_{t_1+u} h_{t_1+u+v}\right)\right) \\ &= c_9 \varphi^{v-1} E_t\left(\varepsilon_{t+s} E_{t_1}\left(h_{t_1+u}^{3/2}\right)\right) = c_9 \varphi^{v-1} E_t\left(\varepsilon_{t+s} h_{t+s+u}^{3/2}\right) = c_9 \varphi^{v-1} \theta_{su}^{(3/2)} \end{aligned}$$

Finally, repeatedly applying the tower law, we get that: $E_t\left(r_{t+s} r_{t+s+u} r_{t+s+u+v} r_{t+s+u+v+w}\right) = \mu^4$.

(e) Centred Moments

The second conditional centred moment of the forward returns (i.e. the conditional variance of the forward return) is:

$$\mu_{r,s}^{(2)} = E_t\left(\varepsilon_{t+s}^2\right) = E_t\left(h_{t+s}\right) = \tilde{\mu}_{h,s}^{(1)} = \bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right),$$

and for the aggregated returns it is:

$$M_{r,n}^{(2)} = E_t\left(\left(\sum_{s=1}^{n} \varepsilon_{t+s}\right)^2\right) = \sum_{s=1}^{n} \tilde{\mu}_{h,s}^{(1)} + 2\sum_{s=1}^{n}\sum_{u=1}^{n-s} \underbrace{E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\right)}_{0} = \sum_{s=1}^{n} \tilde{\mu}_{h,s}^{(1)}$$

Hence, $$M_{r,n}^{(2)} = n\bar{h} + (1-\varphi)^{-1}\left(1-\varphi^n\right)\left(h_{t+1} - \bar{h}\right).$$

The third conditional centred moment of the forward returns is:

$$\mu_{r,s}^{(3)} = E_t\left(\varepsilon_{t+s}^3\right) = \tau_z E_t\left(h_{t+s}^{3/2}\right) \simeq \tfrac{1}{8}\tau_z\left(5\left(\mu_{h,s}^{(1)}\right)^{3/2} + 3\,\mu_{h,s}^{(2)}\left(\mu_{h,s}^{(1)}\right)^{-1/2}\right),$$

so in the special cases when the innovation distribution is either the standard normal, or the standardized Student *t*, $\mu_{r,s}^{(3)} = 0$. The third conditional centred moment of the aggregated returns is:

$$M_{r,n}^{(3)} = E_t\left(\left(\sum_{s=1}^{n}\varepsilon_{t+s}\right)^3\right) = \sum_{s=1}^{n} E_t\left(\varepsilon_{t+s}^3\right) + 3\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left[E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}\right) + E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right)\right]$$
$$+6\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u} E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}\right)$$
$$= \tau_z\sum_{s=1}^{n} E_t\left(h_{t+s}^{3/2}\right) + 3\sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right)$$
$$\simeq \tfrac{1}{8}\tau_z\sum_{s=1}^{n}\left(5\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3/2} + 3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-1/2}\right) + 3\sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right);$$

for the normal and Student *t* GJR models, this expression simplifies to: $M_{r,n}^{(3)} = 3\sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right)$.

The fourth conditional centred moment of the forward returns is:

$$\mu_{r,s}^{(4)} = E_t\left(\varepsilon_{t+s}^4\right) = \kappa_z\tilde{\mu}_{h,s}^{(2)},$$

so in the special case that the innovation distribution is the standard normal, $\mu_{r,s}^{(4)} = 3\tilde{\mu}_{h,s}^{(2)}$, while for the Student *t* GJR we get: $\mu_{r,s}^{(4)} = 3\frac{\nu-2}{\nu-4}\tilde{\mu}_{h,s}^{(2)}$.

The fourth conditional centred moment of the aggregated returns is:

$$M_{r,n}^{(4)} = E_t\left(\left(\sum_{s=1}^{n}\varepsilon_{t+s}\right)^4\right)$$
$$= \sum_{s=1}^{n}\varepsilon_{t+s}^4 + \sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(4\left(E_t\left(\varepsilon_{t+s}^3\varepsilon_{t+s+u}\right) + E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^3\right)\right) + 6E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right)\right)$$
$$+12\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u} E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}\right) + E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\varepsilon_{t+s+u+v}\right) + E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}^2\right)$$
$$+24\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\sum_{w=1}^{n-s-u-v} E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}\varepsilon_{t+s+u+v+w}\right)$$
$$= \kappa_z\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)} + \sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(4E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^3\right) + 6E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right)\right) + 12\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u} E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}^2\right)$$

In the special case that the conditional distribution is the standard normal,

$$M_{r,n}^{(4)} = 3\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)} + 6\sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right) + 12\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u} E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}^2\right),$$

while for a Student *t* GJR we obtain,

$$M_{r,n}^{(4)} = 3\frac{\nu-2}{\nu-4}\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)} + 6\sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right) + 12\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u} E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}^2\right).$$

(f) Standardized Moments

The skewness of the forward returns is:

$$\tau_{r,s} = \mu_{r,s}^{(3)}\left(\mu_{r,s}^{(2)}\right)^{-3/2} = \tau_z E_t\left(h_{t+s}^{3/2}\right)\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-3/2} \simeq \tfrac{1}{8}\tau_z\left(5\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3/2} + 3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-1/2}\right)\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-3/2}$$
$$= \tfrac{1}{8}\tau_z\left(5 + 3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-2}\right)$$

It can be easily observed that if we used only a first order Taylor series expansion we would obtain $\tau_{r,s} \approx \tau_z$ and that $\tau_{r,s} = 0$ for both the normal and Student *t* GJR.

The kurtosis of the forward returns is:

$$\kappa_{r,s} = \mu_{r,s}^{(4)} \left(\mu_{r,s}^{(2)}\right)^{-2} = \kappa_z \tilde{\mu}_{h,s}^{(2)} \left(\tilde{\mu}_{h,s}^{(1)}\right)^{-2},$$

so in the special cases that the conditional distribution is standard normal $\kappa_{h,s} = 3\tilde{\mu}_{h,s}^{(2)} \left(\tilde{\mu}_{h,s}^{(1)}\right)^{-2}$, while for the Student *t* GJR we obtain: $\kappa_{h,s} = 3\frac{\nu-2}{\nu-4}\tilde{\mu}_{h,s}^{(2)} \left(\tilde{\mu}_{h,s}^{(1)}\right)^{-2}$.

Finally, the skewness and kurtosis of the aggregated returns are:

$$\mathrm{T}_{r,n} = M_{r,n}^{(3)} \left(M_{r,n}^{(2)}\right)^{-3/2} \text{ and } \mathrm{K}_{r,n} = M_{r,n}^{(4)} \left(M_{r,n}^{(2)}\right)^{-2}.$$

A.2: Conditional Moments of Forward and Aggregated Variances

(a) First Moments

Applying the conditional expectation operator to (17), the first un-centred conditional moment of the forward variance may be written:

$$\tilde{\mu}_{h,s}^{(1)} = \mu_{r,s}^{(2)} = \bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right)$$

Similarly, the first un-centred conditional moment of the aggregated variance becomes:

$$\tilde{M}_{h,n}^{(1)} = \sum_{s=1}^{n} \tilde{\mu}_{h,s}^{(1)} = n\bar{h} + (1-\varphi)^{-1}(1-\varphi^n)\left(h_{t+1} - \bar{h}\right)$$

or equivalently, in recursive form: $\tilde{M}_{h,n}^{(1)} = \tilde{M}_{h,n-1}^{(1)} + \bar{h} + \varphi^{n-1}\left(h_{t+1} - \bar{h}\right)$.

(b) Second Moments

The second moment of the forward variance is:

$$\begin{aligned}
\tilde{\mu}_{h,s}^{(2)} &= E_t\left(\left(\omega + \left(\alpha + \lambda I_{t+s-1}^{-}\right)\varepsilon_{t+s-1}^2 + \beta h_{t+s-1}\right)^2\right) \\
&= \omega^2 + 2\omega\varphi\tilde{\mu}_{h,s-1}^{(1)} + \left(\varphi^2 + (\kappa_z - 1)(\alpha + \lambda F_0)^2 + \kappa_z\lambda^2 F_0(1 - F_0)\right)\tilde{\mu}_{h,s-1}^{(2)} \\
&= \sum_{i=1}^{s-1} \gamma^{i-1}\left(\omega^2 + 2\omega\varphi\left(\bar{h} + \varphi^{s-i-1}\left(h_{t+1} - \bar{h}\right)\right)\right) + \gamma^{s-1}h_{t+1}^2.
\end{aligned}$$

When $\gamma = 1$, the expression for the second moment of the forward variance becomes:

$$\tilde{\mu}_{h,s}^{(2)} = (s-1)\left(\omega^2 + 2\omega\varphi\bar{h}\right) + 2\varphi\bar{h}\left(1 - \varphi^{s-1}\right)\left(h_{t+1} - \bar{h}\right) + h_{t+1}^2$$

For $\gamma \neq 1$ (and $\gamma \neq \varphi$), we introduce the following additional notation:

$$c_1 = \left(\omega^2 + 2\omega\varphi\bar{h}\right)(1-\gamma)^{-1}, \quad c_2 = 2\omega\varphi\left(h_{t+1} - \bar{h}\right)(\varphi - \gamma)^{-1} \text{and} \quad c_3 = c_1 + c_2.$$

Now the expression for the second moment of variance may be written:

$$\tilde{\mu}_{h,s}^{(2)} = c_1 + \left(h_{t+1}^2 - c_3\right)\gamma^{s-1} + c_2\varphi^{s-1}.$$

The second moment of the aggregated variance is given by:

$$\tilde{M}_{h,n}^{(2)} = E_t\left[\left(\sum_{s=1}^{n} h_{t+s}\right)^2\right] = \sum_{s=1}^{n} \tilde{\mu}_{h,s}^{(2)} + 2\sum_{s=1}^{n}\sum_{u=1}^{n-s} \tilde{\mu}_{h,su}^{(1,1)}$$

$$\begin{aligned}\tilde{\mu}^{(1,1)}_{h,su} &= E_t\left(h_{t+s}\left(\omega+\left(\alpha+\lambda I^-_{t+s+u-1}\right)\varepsilon^2_{t+s+u-1}+\beta h_{t+s+u-1}\right)\right)=\omega\tilde{\mu}^{(1)}_{h,s}+\varphi\tilde{\mu}^{(1,1)}_{h,s(u-1)}\\ &= \bar{h}\tilde{\mu}^{(1)}_{h,s}+\varphi^u\left(\tilde{\mu}^{(2)}_{h,s}-\bar{h}\tilde{\mu}^{(1)}_{h,s}\right),\end{aligned}$$

hence

$$\tilde{M}^{(2)}_{h,n}=\sum_{s=1}^{n}\left(\tilde{\mu}^{(2)}_{h,s}+2\bar{h}\left(n-s\right)\tilde{\mu}^{(1)}_{h,s}\right)+2\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\varphi^u\left(\tilde{\mu}^{(2)}_{h,s}-\bar{h}\tilde{\mu}^{(1)}_{h,s}\right)\right). \tag{20}$$

Consider the first sum (20). For $\gamma\neq 1$ and $\gamma\neq\varphi$, we have:

$$\begin{aligned}\sum_{s=1}^{n}\tilde{\mu}^{(2)}_{h,s} &= \sum_{s=1}^{n}\left(c_1+\left(h^2_{t+1}-c_3\right)\gamma^{s-1}+c_2\varphi^{s-1}\right)\\ &= nc_1+\left(h^2_{t+1}-c_3\right)(1-\gamma)^{-1}\left(1-\gamma^n\right)+c_2(1-\varphi)^{-1}\left(1-\varphi^n\right),\end{aligned} \tag{21}$$

and

$$\begin{aligned}\sum_{s=1}^{n}\left(n-s\right)\tilde{\mu}^{(1)}_{h,s} &= \sum_{s=1}^{n}\left(n-s\right)\left(\bar{h}+\varphi^{s-1}\left(h_{t+1}-\bar{h}\right)\right)\\ &= \tfrac{n(n-1)}{2}\bar{h}+(1-\varphi)^{-1}\left[n-(1-\varphi)^{-1}\left(1-\varphi^n\right)\right]\left(h_{t+1}-\bar{h}\right)\end{aligned} \tag{22}$$

Next we evaluate the double sum term in (20). We have:

$$\begin{aligned}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\varphi^u\tilde{\mu}^{(2)}_{h,s} &= \varphi(1-\varphi)^{-1}\sum_{s=1}^{n}\tilde{\mu}^{(2)}_{h,s}\left(1-\varphi^{n-s}\right)\\ &= \varphi(1-\varphi)^{-1}\left[\begin{array}{l} n\left(c_1-c_2\varphi^{n-1}\right)+\left(c_2-c_1\right)(1-\varphi)^{-1}\left(1-\varphi^n\right)+\left(h^2_{t+1}-c_3\right)\\ \left[(1-\gamma)^{-1}\left(1-\gamma^n\right)-(\varphi-\gamma)^{-1}\left(\varphi^n-\gamma^n\right)\right]\end{array}\right]\end{aligned}$$

and

$$\begin{aligned}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\varphi^u\tilde{\mu}^{(1)}_{h,s} &= \varphi(1-\varphi)^{-1}\sum_{s=1}^{n}\left(1-\varphi^{n-s}\right)\left(\bar{h}+\varphi^{s-1}\left(h_{t+1}-\bar{h}\right)\right)\\ &= \varphi(1-\varphi)^{-1}\left(n\left(\bar{h}-\varphi^{n-1}\left(h_{t+1}-\bar{h}\right)\right)+(1-\varphi)^{-1}\left(h_{t+1}-2\bar{h}\right)\left(1-\varphi^n\right)\right).\end{aligned} \tag{23}$$

Thus the final expression for $\tilde{M}^{(2)}_{h,n}$ is:

$$\begin{aligned}\tilde{M}^{(2)}_{h,n} &= nc_1+\left(h^2_{t+1}-c_3\right)(1-\gamma)^{-1}\left(1-\gamma^n\right)+c_2(1-\varphi)^{-1}\left(1-\varphi^n\right)\\ &+2\bar{h}\left(\tfrac{n(n-1)}{2}\bar{h}+(1-\varphi)^{-1}\left[n-(1-\varphi)^{-1}\left(1-\varphi^n\right)\right]\left(h_{t+1}-\bar{h}\right)\right)\\ &+2\varphi(1-\varphi)^{-1}\left[\begin{array}{l} n\left(c_1-c_2\varphi^{n-1}\right)+\left(c_2-c_1\right)(1-\varphi)^{-1}\left(1-\varphi^n\right)+\left(h^2_{t+1}-c_3\right)\\ \left[(1-\gamma)^{-1}\left(1-\gamma^n\right)-(\varphi-\gamma)^{-1}\left(\varphi^n-\gamma^n\right)\right]\end{array}\right]\\ &-2\bar{h}\varphi(1-\varphi)^{-1}\left(n\left(\bar{h}-\varphi^{n-1}\left(h_{t+1}-\bar{h}\right)\right)+(1-\varphi)^{-1}\left(h_{t+1}-2\bar{h}\right)\left(1-\varphi^n\right)\right).\end{aligned} \tag{24}$$

(c) Third Moments

We now consider the third moment of the forward variance:

$$
\begin{aligned}
\tilde{\mu}_{h,s}^{(3)} &= E_t\left[\left(\omega + \left(\alpha + \lambda I_{t+s-1}^{-}\right)\varepsilon_{t+s-1}^2 + \beta h_{t+s-1}\right)^3\right] \\
&= \omega^3 + 3\omega^2\varphi\tilde{\mu}_{h,s-1}^{(1)} + 3\omega\underbrace{\left[\kappa_z\left(\alpha^2 + \lambda\left(2\alpha+\lambda\right)F_0\right) + \beta^2 + 2\beta\left(\alpha + \lambda F_0\right)\right]}_{\gamma}\tilde{\mu}_{h,s-1}^{(2)} \\
&+ \left[\mu_z^{(6)}\left(\alpha^3 + 3\alpha\lambda\left(\alpha+\lambda\right)F_0 + \lambda^3 F_0\right) + 3\beta\kappa_z\left(\alpha^2 + \lambda\left(2\alpha+\lambda\right)F_0\right) + 3\beta^2\left(\alpha + \lambda F_0\right) + \beta^3\right]\tilde{\mu}_{h,s-1}^{(3)} \\
&= \sum_{i=0}^{s-2} c_4^i\left(\omega^3 + 3\omega^2\varphi\tilde{\mu}_{h,s-i-1}^{(1)} + 3\omega\gamma\tilde{\mu}_{h,s-i-1}^{(2)}\right) + c_4^{s-1}h_{t+1}^3,
\end{aligned}
$$

where $c_4 = \mu_z^{(6)}\left(\alpha^3 + 3\alpha\lambda\left(\alpha+\lambda\right)F_0 + \lambda^3F_0\right) + 3\beta\gamma - \beta^2\left(2\beta + 3\left(\alpha + \lambda F_0\right)\right)$.

For the special case when innovations are normally distributed, $F_0 = \frac{1}{2}$ and $\mu_z^{(6)} = 15$. Similarly, when innovations are Student *t* distributed, $F_0 = \frac{1}{2}$ still and $\mu_z^{(6)} = 15\frac{(\nu-2)^2}{(\nu-4)(\nu-6)}$.

For $c_4 \neq 1$, we get that:

$$
\begin{aligned}
\tilde{\mu}_{h,s}^{(3)} &= \omega\left(\omega^2 + 3\omega\varphi\bar{h} + 3\gamma c_1\right)\left(1 - c_4\right)^{-1} + \left(3\omega^2\varphi\left(h_{t+1} - \bar{h}\right) + 3\omega\gamma c_2\right)\left(\varphi - c_4\right)^{-1}\varphi^{s-1} \\
&+ c_{18}c_4^{s-1} + 3\omega\gamma\left(-c_3 + h_{t+1}^2\right)\left(\gamma - c_4\right)^{-1}\gamma^{s-1}.
\end{aligned}
$$

where

$$
c_{18} = h_{t+1}^3 - \omega\left(\omega^2 + 3\omega\varphi\bar{h} + 3\gamma c_1\right)\left(1 - c_4\right)^{-1} - \left(3\omega^2\varphi\left(h_{t+1} - \bar{h}\right) + 3\omega\gamma\left(-c_1 + h_{t+1}^2\right)\right)\left(\varphi - c_4\right)^{-1}
$$

For $c_4 = 1$, the expression for $\tilde{\mu}_{h,s}^{(3)}$ becomes:

$$
\begin{aligned}
\tilde{\mu}_{h,s}^{(3)} &= \sum_{i=0}^{s-2}\left(\omega^3 + 3\omega^2\varphi\tilde{\mu}_{h,s-i-1}^{(1)} + 3\omega\gamma\tilde{\mu}_{h,s-i-1}^{(2)}\right) + h_{t+1}^3 \\
&= h_{t+1}^3 + (s-1)\,\omega\left(\omega^2 + 3\bar{h}\varphi\omega + 3\gamma c_1\right) + 3\omega\left[\omega\varphi\left(h_{t+1} - \bar{h}\right) + \gamma c_2\right]\left(1 - \varphi\right)^{-1}\left(1 - \varphi^{s-1}\right) \\
&+ 3\omega\gamma\left(h_{t+1}^2 - c_3\right)\left(1 - \gamma\right)^{-1}\left(1 - \gamma^{s-1}\right).
\end{aligned}
$$

For the third moment of aggregated variance we write:

$$
\begin{aligned}
\tilde{M}_{h,n}^{(3)} &= E_t\left(\left(\sum_{s=1}^{n} h_{t+s}\right)^3\right) = \sum_{s=1}^{n} E_t\left(h_{t+s}^3\right) + 3\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left[E_t\left(h_{t+s}^2 h_{t+s+u}\right) + E_t\left(h_{t+s}h_{t+s+u}^2\right)\right] \\
&+ 6\sum_{s=1}^{n}\sum_{u=1}^{T-s}\sum_{v=1}^{T-s-u} E_t\left(h_{t+s}h_{t+s+u}h_{t+s+u+v}\right) \\
&= \sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(3)} + 3\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\tilde{\mu}_{h,su}^{(1,2)} + \tilde{\mu}_{h,su}^{(2,1)}\right) + 6\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\tilde{\mu}_{h,suv}^{(1,1,1)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mu}_{h,su}^{(2,1)} &= E_t\left(h_{t+s}^2\left(\omega + \left(\alpha + \lambda I_{t+s+u-1}^{-}\right)\varepsilon_{t+s+u-1}^2 + \beta h_{t+s+u-1}\right)\right) = \omega\tilde{\mu}_{h,s}^{(2)} + \varphi\tilde{\mu}_{h,s(u-1)}^{(2,1)} \\
&= \bar{h}\tilde{\mu}_{h,s}^{(2)} + \varphi^u\left(\tilde{\mu}_{h,s}^{(3)} - \bar{h}\tilde{\mu}_{h,s}^{(2)}\right).
\end{aligned} \tag{25}
$$

$$\tilde{\mu}_{h,su}^{(1,2)} = E_t\left(h_{t+s}\left(\begin{array}{l}\omega^2+\left(\alpha^2+2\alpha\lambda I_{t+s+u-1}^{-}+\lambda^2 I_{t+s+u-1}^{-}\right)\varepsilon_{t+s+u-1}^4+\beta^2 h_{t+s+u-1}^2\\+2\omega\left(\alpha+\lambda I_{t+s+u-1}^{-}\right)\varepsilon_{t+s+u-1}^2+2\omega\beta h_{t+s+u-1}\\+2\beta\left(\alpha+\lambda I_{t+s+u-1}^{-}\right)\varepsilon_{t+s+u-1}^2 h_{t+s+u-1}\end{array}\right)\right)$$
$$= \omega^2\tilde{\mu}_{h,s}^{(1)}+2\omega\varphi\tilde{\mu}_{h,s(u-1)}^{(1,1)}+\gamma\tilde{\mu}_{h,s(u-1)}^{(1,2)}. = \sum_{j=0}^{u-1}\gamma^j\left(\omega^2\tilde{\mu}_{h,s}^{(1)}+2\omega\varphi\tilde{\mu}_{h,s(u-j-1)}^{(1,1)}\right)+\gamma^u\tilde{\mu}_{h,s}^{(3)}. \qquad (26)$$

$\tilde{\mu}_{h,suv}^{(1,1,1)} = E_t\left(h_{t+s}E_{t+s}\left(h_{t+s+u}h_{t+s+u+v}\right)\right) = E_t\left(h_{t+s}E_{t_1}\left(h_{t_1+u}h_{t_1+u+v}\right)\right)$, where $t_1 = t+s$.

We have already shown that $\tilde{\mu}_{h,su}^{(1,1)} = E_t\left(h_{t+s}h_{t+s+u}\right) = \bar{h}\tilde{\mu}_{h,s}^{(1)}+\varphi^u\left(\tilde{\mu}_{h,s}^{(2)}-\bar{h}\tilde{\mu}_{h,s}^{(1)}\right)$, thus:

$E_{t_1}\left(h_{t_1+u}h_{t_1+u+v}\right) = \bar{h}\tilde{\tilde{\mu}}_{h,u}^{(1)}+\varphi^v\left(\tilde{\tilde{\mu}}_{h,u}^{(2)}-\bar{h}\tilde{\tilde{\mu}}_{h,u}^{(1)}\right)$, where $\tilde{\tilde{\mu}}_{h,u}^{(1)} = E_{t_1}\left(h_{t_1+u}\right)$ and $\tilde{\tilde{\mu}}_{h,u}^{(2)} = E_{t_1}\left(h_{t_1+u}^2\right)$.

Since $\bar{h}\tilde{\mu}_{h,s}^{(1)}+\varphi^u\left(\tilde{\mu}_{h,s}^{(2)}-\bar{h}\tilde{\mu}_{h,s}^{(1)}\right) = \left(\varphi^u\gamma^{s-1}\right)h_{t+1}^2+\left(\bar{h}\varphi^{s-1}\left(1-\varphi^u\right)+2\omega\varphi^{u+1}(\varphi-\gamma)^{-1}\left(\varphi^{s-1}-\gamma^{s-1}\right)\right)h_{t+1}+$ $\bar{h}^2\left(1-\varphi^{s-1}\right) + \varphi^u\left(c_1+\left(2\omega\varphi\bar{h}(\varphi-\gamma)^{-1}-c_1\right)\gamma^{s-1}-2\omega\varphi^s\bar{h}(\varphi-\gamma)^{-1}-\bar{h}^2\left(1-\varphi^{s-1}\right)\right)$, we get:

$$E_{t_1}\left(h_{t_1+u}h_{t_1+u+v}\right) = \left(\varphi^v\gamma^{u-1}\right)h_{t+s+1}^2+\left(\bar{h}\varphi^{u-1}\left(1-\varphi^v\right)+2\omega\varphi^{v+1}(\varphi-\gamma)^{-1}\left(\varphi^{u-1}-\gamma^{u-1}\right)\right)h_{t+s+1}$$
$$+\bar{h}^2\left(1-\varphi^{u-1}\right)+\varphi^v\left(\begin{array}{l}c_1+\left(2\omega\varphi\bar{h}(\varphi-\gamma)^{-1}-c_1\right)\gamma^{u-1}\\-2\omega\varphi^u\bar{h}(\varphi-\gamma)^{-1}-\bar{h}^2\left(1-\varphi^{u-1}\right)\end{array}\right),$$

and the expression for $\tilde{\mu}_{h,suv}^{(1,1,1)}$ now becomes:

$$\tilde{\mu}_{h,suv}^{(1,1,1)} = \left(\varphi^v\gamma^{u-1}\right)E_t\left(h_{t+s}h_{t+s+1}^2\right)$$
$$+\left(\bar{h}\varphi^{u-1}\left(1-\varphi^v\right)+2\omega\varphi^{v+1}(\varphi-\gamma)^{-1}\left(\varphi^{u-1}-\gamma^{u-1}\right)\right)E_t\left(h_{t+s}h_{t+s+1}\right)$$
$$+\left[\bar{h}^2\left(1-\varphi^{u-1}\right)+\varphi^v\left(\begin{array}{l}c_1+\left(2\omega\varphi\bar{h}(\varphi-\gamma)^{-1}-c_1\right)\gamma^{u-1}\\-2\omega\varphi^u\bar{h}(\varphi-\gamma)^{-1}-\bar{h}^2\left(1-\varphi^{u-1}\right)\end{array}\right)\right]\tilde{\mu}_{h,s}^{(1)}.$$

But $E_t\left(h_{t+s}h_{t+s+1}^2\right) = E_t\left(h_{t+s}\left(\omega+\left(\alpha+\lambda I_{t+s}^{-}\right)\varepsilon_{t+s}^2+\beta h_{t+s}\right)^2\right) = \omega^2\tilde{\mu}_{h,s}^{(1)}+2\omega\varphi\tilde{\mu}_{h,s}^{(2)}+\gamma\tilde{\mu}_{h,s}^{(3)}$ and $E_t\left(h_{t+s}h_{t+s+1}\right) = E_t\left(h_{t+s}\left(\omega+\left(\alpha+\lambda I_{t+s}^{-}\right)\varepsilon_{t+s}^2+\beta h_{t+s}\right)\right) = \omega\tilde{\mu}_{h,s}^{(1)}+\varphi\tilde{\mu}_{h,s}^{(2)}$. Hence the final expression for $\tilde{\mu}_{h,suv}^{(1,1,1)}$ is:

$$\tilde{\mu}_{h,suv}^{(1,1,1)} = \left(\varphi^v\gamma^{u-1}\right)\left(\omega^2\tilde{\mu}_{h,s}^{(1)}+2\omega\varphi\tilde{\mu}_{h,s}^{(2)}+\gamma\tilde{\mu}_{h,s}^{(3)}\right)$$
$$+\left(\bar{h}\varphi^{u-1}\left(1-\varphi^v\right)+2\omega\varphi^{v+1}(\varphi-\gamma)^{-1}\left(\varphi^{u-1}-\gamma^{u-1}\right)\right)\left(\omega\tilde{\mu}_{h,s}^{(1)}+\varphi\tilde{\mu}_{h,s}^{(2)}\right)$$
$$+\left[\bar{h}^2\left(1-\varphi^{u-1}\right)+\varphi^v\left(\begin{array}{l}c_1+\left(2\omega\varphi\bar{h}(\varphi-\gamma)^{-1}-c_1\right)\gamma^{u-1}\\-2\omega\varphi^u\bar{h}(\varphi-\gamma)^{-1}-\bar{h}^2\left(1-\varphi^{u-1}\right)\end{array}\right)\right]\tilde{\mu}_{h,s}^{(1)}.$$

(d) Fourth Moments

For the fourth moment of the forward variance we write:

$$\tilde{\mu}^{(4)}_{h,s} = E_t\left[\left(\omega + \left(\alpha + \lambda I^-_{t+s-1}\right)\varepsilon^2_{t+s-1} + \beta h_{t+s-1}\right)^4\right] = \omega^4 + 4\omega^3\varphi\tilde{\mu}^{(1)}_{h,s-1} + c_5\tilde{\mu}^{(2)}_{h,s-1} + c_6\tilde{\mu}^{(3)}_{h,s-1} + c_7\tilde{\mu}^{(4)}_{h,s-1}$$
$$= \sum_{j=0}^{s-2} c_7^j\left(\omega^4 + 4\omega^3\varphi\tilde{\mu}^{(1)}_{h,s-j-1} + c_5\tilde{\mu}^{(2)}_{h,s-j-1} + c_6\tilde{\mu}^{(3)}_{h,s-j-1}\right) + c_7^{s-1}h^4_{t+1}$$

where

$$c_5 = 6\omega^2\left(\kappa_z\left(\alpha^2 + \lambda F_0\left(\lambda + 2\alpha\right)\right) + \beta^2\right) + 12\omega^2\beta\left(\alpha + \lambda F_0\right) = 6\omega^2\gamma$$

$$c_6 = 4\omega\begin{bmatrix}\mu^{(6)}_z\left(\alpha^3 + F_0\left(\lambda^3 + 3\left(\alpha^2\lambda + \alpha\lambda^2\right)\right)\right) \\ +3\kappa_z\beta\left(\alpha^2 + \lambda^2F_0 + 2\alpha\lambda F_0\right) + 3\beta^2\left(\alpha + \lambda F_0\right) + \beta^3\end{bmatrix} = 4\omega c_4$$

$$c_7 = \mu^{(8)}_z\left(\alpha^4 + F_0\left(\lambda^4 + 4\left(\alpha^3\lambda + \alpha\lambda^3\right) + 6\alpha^2\lambda^2\right)\right) + \beta^4$$
$$+4\left[\mu^{(6)}_z\beta\left(\alpha^3 + F_0\left(\lambda^3 + 3\left(\alpha^2\lambda + \alpha\lambda^2\right)\right)\right) + \beta^3\left(\alpha + \lambda F_0\right)\right]$$
$$+6\kappa_z\beta^2\left(\alpha^2 + \lambda^2F_0 + 2\alpha\lambda F_0\right).$$

When the innovations are normally distributed, $\mu^{(8)}_z = 105$, while when they are Student *t* distributed $\mu^{(8)}_z = 105\frac{(\nu-2)^3}{(\nu-4)(\nu-6)(\nu-8)}$. Finally, for the fourth moment of aggrregated variance we write:

$$\tilde{M}^{(4)}_{h,n} = \sum_{s=1}^{n}\tilde{\mu}^{(4)}_{h,s} + \sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(4\left(\tilde{\mu}^{(3,1)}_{h,su} + \tilde{\mu}^{(1,3)}_{h,su}\right) + 6\tilde{\mu}^{(2,2)}_{h,su}\right)$$
$$+12\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\left(\tilde{\mu}^{(2,1,1)}_{h,suv} + \tilde{\mu}^{(1,2,1)}_{h,suv} + \tilde{\mu}^{(1,1,2)}_{h,suv}\right) + 24\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\sum_{w=1}^{n-s-u-v}\tilde{\mu}^{(1,1,1,1)}_{h,suvw}.$$

$$\tilde{\mu}^{(3,1)}_{h,su} = E_t\left(h^3_{t+s}\left(\omega + \left(\alpha + \lambda I^-_{t+s+u-1}\right)\varepsilon^2_{t+s+u-1} + \beta h_{t+s+u-1}\right)\right) = \bar{h}\tilde{\mu}^{(3)}_{h,s} + \varphi^u\left(\tilde{\mu}^{(4)}_{h,s} - \bar{h}\tilde{\mu}^{(3)}_{h,s}\right)$$

$$\tilde{\mu}^{(1,3)}_{h,su} = E_t\left(h_{t+s}\left(\omega + \left(\alpha + \lambda I^-_{t+s+u-1}\right)\varepsilon^2_{t+s+u-1} + \beta h_{t+s+u-1}\right)^3\right)$$
$$= \omega^3\tilde{\mu}^{(1)}_{h,s} + 3\omega\left(\omega\varphi\tilde{\mu}^{(1,1)}_{h,s(u-1)} + \gamma\tilde{\mu}^{(1,2)}_{h,s(u-1)}\right) + c_4\tilde{\mu}^{(1,3)}_{h,s(u-1)}$$
$$= \sum_{j=0}^{u-1} c_4^j\left(\omega^3\tilde{\mu}^{(1)}_{h,s} + 3\omega\left(\omega\varphi\tilde{\mu}^{(1,1)}_{h,s(u-j-1)} + \gamma\tilde{\mu}^{(1,2)}_{h,s(u-j-1)}\right)\right) + c_4^u\tilde{\mu}^{(4)}_{h,s}$$

$$\tilde{\mu}^{(2,2)}_{h,su} = E_t\left(h^2_{t+s}\left(\omega + \left(\alpha + \lambda I^-_{t+s+u-1}\right)\varepsilon^2_{t+s+u-1} + \beta h_{t+s+u-1}\right)^2\right)$$
$$= \omega^2\tilde{\mu}^{(2)}_{h,s} + 2\omega\varphi\tilde{\mu}^{(2,1)}_{h,s(u-1)} + \gamma\tilde{\mu}^{(2,2)}_{h,s(u-1)} = \sum_{j=0}^{u-1}\gamma^j\left(\omega^2\tilde{\mu}^{(2)}_{h,s} + 2\omega\varphi\tilde{\mu}^{(2,1)}_{h,s(u-j-1)}\right) + \gamma^u\tilde{\mu}^{(4)}_{h,s}.$$

$\tilde{\mu}^{(2,1,1)}_{h,suv} = E_t\left(h^2_{t+s}E_{t+s}\left(h_{t+s+u}h_{t+s+u+v}\right)\right) = E_t\left(h^2_{t+s}E_{t_1}\left(h_{t_1+u}h_{t_1+u+v}\right)\right)$, where $t_1 = t + s$.

By analogy with $\tilde{\mu}^{(1,1,1)}_{h,suv}$, we obtain the following expression for $\tilde{\mu}^{(2,1,1)}_{h,suv}$:

$$\tilde{\mu}^{(2,1,1)}_{h,suv} = \left(\varphi^v\gamma^{u-1}\right)\left(\omega^2\tilde{\mu}^{(2)}_{h,s} + 2\omega\varphi\tilde{\mu}^{(3)}_{h,s} + \gamma\tilde{\mu}^{(4)}_{h,s}\right)$$
$$+\left(\bar{h}\varphi^{u-1}\left(1-\varphi^v\right) + 2\omega\varphi^{v+1}(\varphi-\gamma)^{-1}\left(\varphi^{u-1} - \gamma^{u-1}\right)\right)\left(\omega\tilde{\mu}^{(2)}_{h,s} + \varphi\tilde{\mu}^{(3)}_{h,s}\right)$$
$$+\left[\bar{h}^2\left(1-\varphi^{u-1}\right) + \varphi^v\begin{pmatrix}c_1 + \left(2\omega\varphi\bar{h}(\varphi-\gamma)^{-1} - c_1\right)\gamma^{u-1} \\ -2\omega\varphi^u\bar{h}(\varphi-\gamma)^{-1} - \bar{h}^2\left(1-\varphi^{u-1}\right)\end{pmatrix}\right]\tilde{\mu}^{(2)}_{h,s}.$$

Using the same idea (the tower law), we can also solve for $\tilde{\mu}^{(1,2,1)}_{h,suv}$, $\tilde{\mu}^{(1,1,2)}_{h,suv}$ and $\tilde{\mu}^{(1,1,1,1)}_{h,suv}$.

(e) Centred Moments

The second centred moment of the forward variance, i.e. the conditional variance of the forward conditional variance is:

$$\mu_{h,s}^{(2)} = E_t\left(\left(h_{t+s} - \tilde{\mu}_{h,s}^{(1)}\right)^2\right) = \tilde{\mu}_{h,s}^{(2)} - \left(\tilde{\mu}_{h,s}^{(1)}\right)^2$$

The second centred moment of the aggregated variance, i.e. the conditional variance of the aggregated conditional variance is:

$$\begin{aligned} M_{h,n}^{(2)} &= E_t\left(\left(\sum_{s=1}^{n}\left(h_{t+s} - \tilde{\mu}_{h,s}^{(1)}\right)\right)^2\right) = \sum_{s=1}^{n}\left(\tilde{\mu}_{h,s}^{(2)} - \left(\tilde{\mu}_{h,s}^{(1)}\right)^2\right) + 2\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\tilde{\mu}_{h,su}^{(1,1)} - \tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s+u}^{(1)}\right) \\ &= \sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)} + 2\sum_{s=1}^{n}\sum_{u=1}^{n-s}\varphi^u\tilde{\mu}_{h,s}^{(2)} - \sum_{s=1}^{n}\left(\tilde{\mu}_{h,s}^{(1)}\right)^2 + 2\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(1-\varphi^u\right)\bar{h}\tilde{\mu}_{h,s}^{(1)} - 2\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s+u}^{(1)} \end{aligned} \tag{27}$$

or

$$M_{h,n}^{(2)} = \tilde{M}_{h,n}^{(2)} - \sum_{s=1}^{n}\left(\tilde{\mu}_{h,s}^{(1)}\right)^2 - 2\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s+u}^{(1)}. \tag{28}$$

For $\gamma \neq 1$, the expression for the second moment of the aggregated variance is given by (24).

$$\begin{aligned} \sum_{s=1}^{n}\left(\tilde{\mu}_{h,s}^{(1)}\right)^2 &= \sum_{s=1}^{n}\left(\bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right)\right)^2 \\ &= n\bar{h}^2 + \left(h_{t+1} - \bar{h}\right)^2(1-\varphi^2)^{-1}\left(1-\varphi^{2n}\right) + 2\bar{h}\left(h_{t+1} - \bar{h}\right)(1-\varphi)^{-1}\left(1-\varphi^n\right) \end{aligned} \tag{29}$$

$$\begin{aligned} \sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s+u}^{(1)} &= \sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right)\right)\left(\bar{h} + \varphi^{s+u-1}\left(h_{t+1} - \bar{h}\right)\right) \\ &= \tfrac{1}{2}n\left(n-1\right)\bar{h}^2 + \bar{h}\left(h_{t+1} - \bar{h}\right)(1-\varphi)^{-1}\left[n - (1-\varphi)^{-1}\left(1-\varphi^n\right)\right] \\ &+ \bar{h}\left(h_{t+1} - \bar{h}\right)(1-\varphi)^{-1}\left[\varphi(1-\varphi)^{-1}\left(1-\varphi^n\right) - n\varphi^n\right] \\ &+ \left(h_{t+1} - \bar{h}\right)^2(1-\varphi)^{-1}\left[\varphi(1-\varphi^2)^{-1}\left(1-\varphi^{2n}\right) - (1-\varphi)^{-1}\varphi^n\left(1-\varphi^n\right)\right] \end{aligned} \tag{30}$$

For $\gamma = 1$, consider the formula in (27). The expressions for the last three sums do not depend on γ and hence remain the same as in the $\gamma \neq 1$ case (see (22), (23), (29) and (30)), while $\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)}$ and $\sum_{s=1}^{n}\sum_{u=1}^{n-s}\varphi^u\tilde{\mu}_{h,s}^{(2)}$ become:

$$\begin{aligned} \sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)} &= \sum_{s=1}^{n}\left[(s-1)\left(\omega^2 + 2\omega\varphi\bar{h}\right) + 2\varphi\bar{h}\left(1-\varphi^{s-1}\right)\left(h_{t+1} - \bar{h}\right) + h_{t+1}^2\right] \\ &= \tfrac{1}{2}n\left(n-1\right)\left(\omega^2 + 2\omega\varphi\bar{h}\right) + 2\varphi\bar{h}\left(h_{t+1} - \bar{h}\right)\left(n - (1-\varphi)^{-1}\left(1-\varphi^n\right)\right) + nh_{t+1}^2 \end{aligned} \tag{31}$$

$$\sum_{s=1}^{n}\sum_{u=1}^{n-s}\varphi^{u}\tilde{\mu}_{h,s}^{(2)}=\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)}\sum_{u=1}^{n-s}\varphi^{u}=\varphi(1-\varphi)^{-1}\left[\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)}-\sum_{s=1}^{n}\varphi^{n-s}\tilde{\mu}_{h,s}^{(2)}\right] \tag{32}$$

$$\begin{aligned}&\sum_{s=1}^{n}\varphi^{n-s}\tilde{\mu}_{h,s}^{(2)}=\sum_{s=1}^{n}\varphi^{n-s}\left[(s-1)\left(\omega^{2}+2\omega\varphi\bar{h}\right)+2\varphi\bar{h}\left(1-\varphi^{s-1}\right)\left(h_{t+1}-\bar{h}\right)+h_{t+1}^{2}\right]\\&=\left(\omega^{2}+2\omega\varphi\bar{h}\right)(1-\varphi)^{-1}\left[n-(1-\varphi)^{-1}\left(1-\varphi^{n}\right)\right]-2n\bar{h}\left(h_{t+1}-\bar{h}\right)\varphi^{n}\\&+\left[2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right)+h_{t+1}^{2}\right](1-\varphi)^{-1}\left(1-\varphi^{n}\right).\end{aligned} \tag{33}$$

The third centred moment of the forward variance is:

$$\mu_{h,s}^{(3)}=E_{t}\left(\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)^{3}\right)=\tilde{\mu}_{h,s}^{(3)}-3\tilde{\mu}_{h,s}^{(2)}\tilde{\mu}_{h,s}^{(1)}+2\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3} \tag{34}$$

and the third centred moment of the aggregated variance is:

$$\begin{aligned}M_{h,n}^{(3)}&=E_{t}\left(\left(\sum_{s=1}^{n}\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)\right)^{3}\right)\\&=\sum_{s=1}^{n}\left(\tilde{\mu}_{h,s}^{(3)}-3\tilde{\mu}_{h,s}^{(2)}\tilde{\mu}_{h,s}^{(1)}+2\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3}\right)+3\sum_{s=1}^{n}\sum_{u=1}^{n-s}\begin{pmatrix}\tilde{\mu}_{h,su}^{(2,1)}+\tilde{\mu}_{h,su}^{(1,2)}+2\left(\tilde{\mu}_{h,s}^{(1)}+\tilde{\mu}_{h,s+u}^{(1)}\right)\\\left(\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s+u}^{(1)}-\tilde{\mu}_{h,su}^{(1,1)}\right)-\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s+u}^{(2)}-\tilde{\mu}_{h,s+u}^{(1)}\tilde{\mu}_{h,s}^{(2)}\end{pmatrix}\\&+6\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\begin{pmatrix}\tilde{\mu}_{h,suv}^{(1,1,1)}-\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,(s+u)v}^{(1,1)}-\tilde{\mu}_{h,(s+u)}^{(1)}\tilde{\mu}_{h,s(u+v)}^{(1,1)}\\-\tilde{\mu}_{h,(s+u+v)}^{(1)}\tilde{\mu}_{h,su}^{(1,1)}+2\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,(s+u)}^{(1)}\tilde{\mu}_{h,(s+u+v)}^{(1)}\end{pmatrix}\end{aligned}$$

The fourth centred moment of the forward variance is:

$$\mu_{h,s}^{(4)}=\tilde{\mu}_{h,s}^{(4)}-4\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s}^{(3)}+6\left(\tilde{\mu}_{h,s}^{(1)}\right)^{2}\tilde{\mu}_{h,s}^{(2)}-3\left(\tilde{\mu}_{h,s}^{(1)}\right)^{4}$$

Finally, the fourth centred moment of the aggregated variance is:

$$\begin{aligned}M_{h,n}^{(4)}&=E_{t}\left(\left(\sum_{s=1}^{n}\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)\right)^{4}\right)=\sum_{s=1}^{n}E_{t}\left(\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)^{4}\right)\\&+\sum_{s=1}^{n}\sum_{u=1}^{n-s}\begin{pmatrix}4\left(E_{t}\left(\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)^{3}\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right)\right)+E_{t}\left(\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right)^{3}\right)\right)\\+6E_{t}\left(\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)^{2}\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right)^{2}\right)\end{pmatrix}\\&+12\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\begin{bmatrix}E_{t}\left(\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)^{2}\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right)\left(h_{t+s+u+v}-\tilde{\mu}_{h,s+u+v}^{(1)}\right)\right)\\+E_{t}\left(\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right)^{2}\left(h_{t+s+u+v}-\tilde{\mu}_{h,s+u+v}^{(1)}\right)\right)\\+E_{t}\left(\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right)\left(h_{t+s+u+v}-\tilde{\mu}_{h,s+u+v}^{(1)}\right)^{2}\right)\end{bmatrix}\\&+24\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\sum_{w=1}^{n-s-u-v}E_{t}\begin{bmatrix}\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right)\\\left(h_{t+s+u+v}-\tilde{\mu}_{h,s+u+v}^{(1)}\right)\left(h_{t+s+u+v+w}-\tilde{\mu}_{h,s+u+v+w}^{(1)}\right)\end{bmatrix}\end{aligned}$$

Performing the necessary algebaraic calculations yields the formula for $M_{h,n}^{(4)}$ from Theorem 2.

The standardized moments (i.e. skewness and kurtosis) of the forward and aggregated variance distributions are now easily obtained from the central moments, as defined in Section 2.1.

A.3: Limits of the Moments of Returns

This appendix derives the limits of the conditional moments of the forward and aggregated returns of the generic GJR model as the time horizon increases. We also outline the results for two important special cases of the generic framework, namely the normal GJR (i.e. $D(0,1)$ is now the standard normal) and also the normal GARCH(1,1) (i.e. $D(0,1)$ is the standard normal and $\lambda = 0$). We only specify the results for these special cases when they differ from the results obtained for the generic model.

In what follows we use the notation defined in the beginning of this Appendix. For the two special cases where $D(0,1)$ is the standard normal, $\tau_z = 0$, $F_0 = \frac{1}{2}$ and $\kappa_z = 3$ and φ and γ become (for the normal GJR):

$$\varphi = \alpha + \frac{\lambda}{2} + \beta \text{ and } \gamma = \varphi^2 + 2\left(\alpha + \frac{\lambda}{2}\right)^2 + \frac{3}{4}\lambda^2 \,. \tag{35}$$

Moreover, for the normal GARCH(1,1), $\lambda = 0$ and the two constants above simplify further:

$$\varphi = \alpha + \beta \text{ and } \gamma = \varphi^2 + 2\alpha^2 = (\alpha+\beta)^2 + 2\alpha^2 \,. \tag{36}$$

We assume $\varphi \in (0,1)$ and $\varphi \neq \gamma$.

(a) Limits of the forward and aggregated variance

Both the forward variance and the aggregated variance limit, expressed in daily units, are equal to the long term variance, which we have denoted by $\bar{h}$. That is,

$$\lim_{s\to\infty} \mu_{r,s}^{(2)} = \lim_{s\to\infty} \tilde{\mu}_{h,s}^{(1)} = \lim_{s\to\infty} \left(\bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right)\right) = \bar{h}$$

$$\lim_{n\to\infty} \frac{M_{r,n}^{(2)}}{n} = \lim_{n\to\infty} \left[\frac{n\bar{h} + (1-\varphi)^{-1}(1-\varphi^n)\left(h_{t+1} - \bar{h}\right)}{n}\right] = \bar{h}.$$

(b) Limits of the forward and aggregated skewness

The forward skewness limit is:

$$\lim_{s\to\infty} \tau_{r,s} = \lim_{s\to\infty} \left[\tfrac{1}{8}\tau_z \left(5 + 3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-2}\right)\right] = \tfrac{1}{8}\tau_z \left(5 + 3\lim_{s\to\infty}\tilde{\mu}_{h,s}^{(2)}\left(\lim_{s\to\infty}\tilde{\mu}_{h,s}^{(1)}\right)^{-2}\right)$$

where $\lim_{s\to\infty}\tilde{\mu}_{h,s}^{(2)} == \begin{cases} c_1 & \text{if} \gamma \in (0,1)\,, \\ \infty & \text{if} \gamma \in [1,\infty)\,, \end{cases}$ Hence:

$$\lim_{s\to\infty}\tau_{r,s} = \begin{cases} \frac{1}{8}\tau_z\left(5+3\left(\omega^2+2\omega\varphi\bar{h}\right)(1-\gamma)^{-1}\left(\bar{h}\right)^{-2}\right) & \text{if } \gamma\in(0,1)\,, \\ \operatorname{sgn}(\tau_z)\,\infty & \text{if } \gamma\in[1,\infty)\,. \end{cases}$$

For the normal GJR and the normal GARCH(1,1), $\tau_{r,s} = \lim_{s\to\infty}\tau_{r,s} = 0$.

For the limit of the aggregated skewness set:

$$c_{12} = \tfrac{1}{8}\left(\tau_z + 3\left(\alpha\tau_z + \lambda\int_{x=-\infty}^{0} x^3 f(x)\,dx\right)(1-\varphi)^{-1}\right)$$

$$c_{13} = \tfrac{3}{8}\left(\left(\alpha\tau_z + \lambda\int_{x=-\infty}^{0} x^3 f(x)\,dx\right)(1-\varphi)^{-1}\right) = c_{12} - \tfrac{\tau_z}{8}.$$

We have:

$$\lim_{n\to\infty}\mathrm{T}_{r,n} = \lim_{n\to\infty}\frac{M_{r,n}^{(3)}}{n^{3/2}}\left(\frac{M_{r,n}^{(2)}}{n}\right)^{-3/2} = \lim_{n\to\infty}\frac{M_{r,n}^{(3)}}{n^{3/2}}\left[\lim_{n\to\infty}\left(\frac{M_{r,n}^{(2)}}{n}\right)\right]^{-3/2}$$
$$= c_{13}\bar{h}^{-3/2}\lim_{n\to\infty}\left[n^{-3/2}\sum_{s=1}^{n}\left(\frac{c_{12}}{c_{13}}-\varphi^{n-s}\right)\left(5\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3/2}+3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-1/2}\right)\right],$$

Write $S_n = \frac{c_{13}\sum_{s=1}^{n}\left(\frac{c_{12}}{c_{13}}-\varphi^{n-s}\right)\left(5\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3/2}+3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-1/2}\right)}{n^{3/2}}$ and $L = \lim_{n\to\infty} S_n$

$S_{1,n} = \frac{c_{13}\sum_{s=1}^{n}\left(\frac{c_{12}}{c_{13}}-\varphi^{n-s}\right)\left(5\left(\max_{1\le s\le n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{3/2}+3\tilde{\mu}_{r,s}^{(2)}\left(\min_{1\le s\le n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{-1/2}\right)}{n^{3/2}}$ and $L_1 = \lim_{n\to\infty} S_{1,n}$

$S_{2,n} = \frac{c_{13}\sum_{s=1}^{n}\left(\frac{c_{12}}{c_{13}}-\varphi^{n-s}\right)\left(5\left(\min_{1\le s\le n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{3/2}+3\tilde{\mu}_{r,s}^{(2)}\left(\max_{1\le s\le n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{-1/2}\right)}{n^{3/2}}$ and $L_2 = \lim_{n\to\infty} S_{2,n}$.

Case 1

$c_{14} = \frac{c_{12}}{c_{13}} \le 0$, then $c_{14} - \varphi^{n-s} \le 0$ for any s. Also, if $\tau_z < 0$, then $S_{1,n} \le S_n \le S_{2,n}$ for any s.[24]

Hence, if $L_1 = L_2$ then $L = L_1 = L_2$, by the squeeze theorem. We now prove that $L_1 = L_2$. Setting $L_{\max} = \lim_{n\to\infty}\left(\max_{1\le s\le n}\tilde{\mu}_{h,s}^{(1)}\right) = \max\left(\bar{h}, h_{t+1}\right)$ and $L_{\min} = \lim_{n\to\infty}\left(\min_{1\le s\le n}\tilde{\mu}_{h,s}^{(1)}\right) = \min\left(\bar{h}, h_{t+1}\right)$ we may write:

$$L_1 = c_{13}\left[5L_{\max}^{3/2}\lim_{n\to\infty}\left(n^{-1/2}c_{14} - n^{-3/2}(1-\varphi)^{-1}(1-\varphi^n)\right) + 3L_{\min}^{-1/2}\lim_{n\to\infty}n^{-3/2}\sum_{s=1}^{n}\left(c_{14}-\varphi^{n-s}\right)\tilde{\mu}_{h,s}^{(2)}\right]$$

and the first term above is zero. For $\gamma \ne 1$,

$$\sum_{s=1}^{n}\left(c_{14}-\varphi^{n-s}\right)\tilde{\mu}_{h,s}^{(2)} = \sum_{s=1}^{n}\left(c_{14}-\varphi^{n-s}\right)\left(c_1 + \left(h_{t+1}^2 - c_3\right)\gamma^{s-1} + c_2\varphi^{s-1}\right)$$
$$= nc_1c_{14} + c_{14}\left(h_{t+1}^2 - c_3\right)(1-\gamma)^{-1}(1-\gamma^n) + \left(c_2c_{14}-c_1\right)(1-\varphi)^{-1}(1-\varphi^n)$$
$$-\left(h_{t+1}^2 - c_3\right)(\varphi-\gamma)^{-1}(\varphi^n-\gamma^n) - nc_2\varphi^{n-1}.$$

[24] If $\tau_z > 0$, then $S_{2,n} \le S_n \le S_{1,n}$. However, the limit does not change: the proof above still applies, only that $S_{2,n}$ and $S_{1,n}$ swap place above.

For $\gamma = 1$,

$$\sum_{s=1}^{n}\left(c_{14}-\varphi^{n-s}\right)\tilde{\mu}_{h,s}^{(2)} = \sum_{s=1}^{n}\left(c_{14}-\varphi^{n-s}\right)\left[\begin{array}{l}(s-1)\left(\omega^2+2\omega\varphi\bar{h}\right)\\ +2\varphi\bar{h}\left(1-\varphi^{s-1}\right)\left(h_{t+1}-\bar{h}\right)+\tilde{\mu}_{h,1}^{(2)}\end{array}\right]$$

$$= {}^1\!/\!_2 n(n-1)\,c_{14}\left(\omega^2+2\omega\varphi h_0\right)\bar{h} - \varphi\left(\omega^2+2\omega\varphi\bar{h}\right)(1-\varphi)^{-1}$$

$$\left[(1-\varphi)^{-1}(1-\varphi^n)-\varphi^{-1}(n-1)\right]+2c_{14}\varphi h_0\left(h_{t+1}-\bar{h}\right)\left(n-(1-\varphi)^{-1}(1-\varphi^n)\right)$$

$$-2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right)\left[(1-\varphi)^{-1}(1-\varphi^n)-n\varphi^{n-1}\right]+\tilde{\mu}_{h,1}^{(2)}\left(nc_{14}-(1-\varphi)^{-1}(1-\varphi^n)\right).$$

Hence

$$L_1 = \begin{cases} 0 & \text{if } \gamma\in(0,1)\,,\\ \operatorname{sgn}(c_{13})\operatorname{sgn}\left(c_{14}\left(\omega^2+2\omega\varphi\bar{h}\right)\right)\infty & \text{if } \gamma=1,\\ \operatorname{sgn}(c_{13})\operatorname{sgn}\left[\left((\varphi-\gamma)^{-1}-c_{14}(1-\gamma)^{-1}\right)\left(h_{t+1}^2-c_3\right)\right]\infty & \text{if } \gamma\in(1,\infty)\,. \end{cases}$$

Now, for $\gamma\in[1,\infty)$, $\operatorname{sgn}\left(h_{t+1}^2-c_3\right)=1$, $\operatorname{sgn}\left((\varphi-\gamma)^{-1}-c_{14}(1-\gamma)^{-1}\right)=-1$ and $\operatorname{sgn}\left(\omega^2+2\omega\varphi h_0\right)=1$, so $L_1 = \begin{cases} 0 & \text{if}\gamma\in(0,1)\,,\\ \operatorname{sgn}(c_{12})\infty & \text{if}\gamma\in[1,\infty)\,.\end{cases}$

Analogously it can be shown that L_2 is the same limit, hence $L_1 = L_2$ and finally:

$$\lim_{n\to\infty}\mathrm{T}_{r,n} = \begin{cases} 0 & \text{if}\gamma\in(0,1)\,,\\ \operatorname{sgn}(c_{12})\infty & \text{if}\gamma\in[1,\infty)\,.\end{cases}$$

<u>Case 2</u>

$c_{14} > 0$: In this case $\exists$ an integer $\tilde{s}\geq 1$ such that $c_{14}-\varphi^{n-s}>0$ for $\forall s>\tilde{s}$ and

$c_{14}-\varphi^{n-s}\leq 0$ for $\forall\ \tilde{s}\geq s\geq 1$. It can be easily seen that if $c_{14}>\varphi$, then $\tilde{s}=0$.

Write: $S_n = S_n^{(1)}+S_n^{(2)}$, $\tilde{S}_{1,n}=S_{1,n}^{(1)}+S_{2,n}^{(2)}$, $\tilde{S}_{2,n}=S_{2,n}^{(1)}+S_{1,n}^{(2)}$ where

$$S_n^{(1)} = \frac{c_{13}\sum_{s=1}^{\tilde{s}}\left(c_{14}-\varphi^{n-s}\right)\left(5\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3/2}+3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-1/2}\right)}{n^{3/2}},\ S_n^{(2)} = \frac{c_{13}\sum_{s=\tilde{s}+1}^{n}\left(c_{14}-\varphi^{n-s}\right)\left(5\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3/2}+3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-1/2}\right)}{n^{3/2}}$$

$$S_{1,n}^{(1)} = \frac{c_{13}\sum_{s=1}^{\tilde{s}}\left(c_{14}-\varphi^{n-s}\right)\left(5\left(\max_{1\leq s\leq n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{3/2}+3\tilde{\mu}_{r,s}^{(2)}\left(\min_{1\leq s\leq n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{-1/2}\right)}{n^{3/2}},$$

$$S_{2,n}^{(2)} = \frac{c_{13}\sum_{s=\tilde{s}+1}^{n}\left(c_{14}-\varphi^{n-s}\right)\left(5\left(\min_{1\leq s\leq n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{3/2}+3\tilde{\mu}_{r,s}^{(2)}\left(\max_{1\leq s\leq n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{-1/2}\right)}{n^{3/2}}$$

$$S_{2,n}^{(1)} = \frac{c_{13}\sum_{s=1}^{\tilde{s}}\left(c_{14}-\varphi^{n-s}\right)\left(5\left(\min_{1\leq s\leq n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{3/2}+3\tilde{\mu}_{r,s}^{(2)}\left(\max_{1\leq s\leq n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{-1/2}\right)}{n^{3/2}}\ \text{and}$$

$$S_{1,n}^{(2)} = \frac{c_{13}\sum_{s=\tilde{s}+1}^{n}\left(c_{14}-\varphi^{n-s}\right)\left(5\left(\max_{1\leq s\leq n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{3/2}+3\tilde{\mu}_{r,s}^{(2)}\left(\min_{1\leq s\leq n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{-1/2}\right)}{n^{3/2}}\,.$$

If we solve further for $S_{1,n}^{(1)}$, we get:

$$S_{1,n}^{(1)} = c_{13}\left[\begin{array}{l} 5\left(\max\limits_{1\le s\le n}\left(\tilde{\mu}_{h,s}^{(1)}\right)\right)^{3/2} n^{-3/2}\left(\tilde{s}c_{14} - \varphi^{n-\tilde{s}}\sum\limits_{s=1}^{\tilde{s}}\varphi^{\tilde{s}-s}\right) \\ +3\left(\min\limits_{1\le s\le n}\left(\tilde{\mu}_{h,s}^{(1)}\right)\right)^{-1/2} n^{-3/2}\left(c_{14}\sum\limits_{s=1}^{\tilde{s}}\tilde{\mu}_{h,s}^{(2)} - \varphi^{n-\tilde{s}}\sum\limits_{s=1}^{\tilde{s}}\varphi^{\tilde{s}-s}\tilde{\mu}_{h,s}^{(2)}\right)\end{array}\right]$$

Define: $f_1^{a,b} = \sum\limits_{s=a}^{b}\varphi^{b-s}$, $f_2^{a,b} = \sum\limits_{s=a}^{b}\tilde{\mu}_{h,s}^{(2)}$ and $f_3^{a,b} = \sum\limits_{s=a}^{b}\varphi^{b-s}\tilde{\mu}_{h,s}^{(2)}$. $S_{1,n}^{(1)}$ becomes:

$$S_{1,n}^{(1)} = c_{13}\left[5\left(\max_{1\le s\le n}\left(\tilde{\mu}_{h,s}^{(1)}\right)\right)^{3/2} n^{-3/2}\left[\tilde{s}c_{14} - \varphi^{n-\tilde{s}}f_1^{1,\tilde{s}}\right] + 3\left(\min_{1\le s\le n}\left(\tilde{\mu}_{h,s}^{(1)}\right)\right)^{-1/2} n^{-3/2}\left[f_2^{1,\tilde{s}} - \varphi^{n-\tilde{s}}f_3^{1,\tilde{s}}\right]\right],$$

where $f_j^{1,\tilde{s}}$ $j = 1, 2, 3$ are all constant w.r.t. n. Thus $\lim\limits_{n\to\infty} S_{1,n}^{(1)} = 0$. Also

$$S_{2,n}^{(2)} = c_{13}\left[\begin{array}{l} 5\left(\min\limits_{1< s\le n}\left(\tilde{\mu}_{h,s}^{(1)}\right)\right)^{3/2} n^{-3/2}\left((n-\tilde{s})\,c_{14} - f_1^{\tilde{s}+1,n}\right) \\ +3\left(\max\limits_{1< s\le n}\left(\tilde{\mu}_{h,s}^{(1)}\right)\right)^{-1/2} n^{-3/2}\left(c_{14}f_2^{\tilde{s}+1,n} - f_3^{\tilde{s}+1,n}\right)\end{array}\right].$$

Now $f_1^{\tilde{s}+1,n} = (1-\varphi)^{-1}\left(1-\varphi^{n-\tilde{s}}\right)$. Also, for $\gamma \neq 1$,

$$\begin{aligned} f_2^{\tilde{s}+1,n} &= \sum_{s=\tilde{s}+1}^{n}\left(c_1 + \left(h_{t+1}^2 - c_3\right)\gamma^{s-1} + c_2\varphi^{s-1}\right) \\ &= (n-\tilde{s})\,c_1 + \left(h_{t+1}^2 - c_3\right)\gamma^{\tilde{s}}(1-\gamma)^{-1}\left(1-\gamma^{n-\tilde{s}}\right) + c_2\varphi^{\tilde{s}}(1-\varphi)^{-1}\left(1-\varphi^{n-\tilde{s}}\right), \\ f_3^{\tilde{s}+1,n} &= \sum_{s=\tilde{s}+1}^{n}\varphi^{n-s}\left(c_1 + \left(h_{t+1}^2 - c_3\right)\gamma^{s-1} + c_2\varphi^{s-1}\right) \\ &= c_1(1-\varphi)^{-1}\left(1-\varphi^{n-\tilde{s}}\right) + \left(h_{t+1}^2 - c_3\right)\gamma^{\tilde{s}}(\varphi-\gamma)^{-1}\left(\varphi^{n-\tilde{s}} - \gamma^{n-\tilde{s}}\right) + c_2\,(n-\tilde{s})\,\varphi^{n-1}. \end{aligned}$$

For $\gamma = 1$,

$$\begin{aligned} f_2^{\tilde{s}+1,n} &= \sum_{s=\tilde{s}+1}^{n}\left((s-1)\left(\omega^2 + 2\omega\varphi\bar{h}\right) + 2\varphi\bar{h}\left(1-\varphi^{s-1}\right)\left(h_{t+1}-\bar{h}\right) + h_{t+1}^2\right) \\ &= \tfrac{1}{2}\left(\omega^2 + 2\omega\varphi\bar{h}\right)(n-\tilde{s})(n-\tilde{s}-1) - 2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right)\varphi^{\tilde{s}}(1-\varphi)^{-1}\left(1-\varphi^{n-\tilde{s}}\right) \\ &\quad + \left(2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right) + h_{t+1}^2\right)(n-\tilde{s}), \\ f_3^{\tilde{s}+1,n} &= \sum_{s=\tilde{s}+1}^{n}\varphi^{n-s}\tilde{\mu}_{h,s}^{(2)} = \sum_{s=\tilde{s}+1}^{n}\varphi^{n-s}\left((s-1)\left(\omega^2 + 2\omega\varphi\bar{h}\right) + 2\varphi h_0\left(1-\varphi^{s-1}\right)\left(h_{t+1}-\bar{h}\right) + h_{t+1}^2\right) \\ &= \left(\omega^2 + 2\omega\varphi\bar{h}\right)(1-\varphi)^{-1}\left(\varphi(1-\varphi)^{-1}\left(1-\varphi^{n-\tilde{s}}\right) - \tilde{s}\varphi^{n-\tilde{s}}\right) \\ &\quad + \left[2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right) + h_{t+1}^2\right](1-\varphi)^{-1}\left(1-\varphi^{n-\tilde{s}}\right) - \left(2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right)\right)(n-\tilde{s})\,\varphi^{n-1}. \end{aligned}$$

Hence:

$$\begin{aligned} \lim_{n\to\infty}\tilde{S}_{1,n} = \lim_{n\to\infty} S_{2,n}^{(2)} &= \begin{cases} 0 & \text{if } \gamma \in (0,1), \\ \operatorname{sgn}(c_{13})\,\infty & \text{if } \gamma = 1. \\ \operatorname{sgn}\left[c_{13}\left(h_{t+1}^2 - c_3\right)\left(-c_{14}(1-\gamma)^{-1} + (\varphi-\gamma)^{-1}\right)\right]\infty & \text{if } \gamma \in (1,\infty), \end{cases} \\ &= \begin{cases} 0 & \text{if } \gamma \in (0,1), \\ \operatorname{sgn}\left(\tau_z\left(\alpha + \frac{\gamma-\varphi}{3}\right) + \lambda\int\limits_{x=-\infty}^{0} x^3 f(x)\,dx\right)\infty & \text{if } \gamma \in [1,\infty) \end{cases} \end{aligned}$$

Analogously it can be shown that $\lim_{n\to\infty}\tilde{S}_{2,n}$ is identical to $\lim_{n\to\infty}\tilde{S}_{1,n}$. Using $\min\left(\tilde{S}_{1,n},\tilde{S}_{2,n}\right)\le S_n\le\max\left(\tilde{S}_{1,n},\tilde{S}_{2,n}\right)$[25] and the squeeze theorem,[26]

$$\lim_{n\to\infty}\mathrm{T}_{r,n}=\begin{cases}0 & \text{if } \gamma\in(0,1)\,,\\ \operatorname{sgn}\left(\tau_z\left(\alpha+\frac{\gamma-\varphi}{3}\right)+\lambda\int\limits_{x=-\infty}^{0}x^3f(x)\,dx\right)\infty & \text{if } \gamma\in[1,\infty)\,.\end{cases}$$

For the normal GJR, $\tau_z=0$ and $\int\limits_{x=-\infty}^{0}x^3f(x)\,dx=-\sqrt{\frac{2}{\pi}}$ hence: $\lim_{n\to\infty}\mathrm{T}_{r,n}=\begin{cases}0 & \text{if } \gamma\in(0,1)\,,\\ -\operatorname{sgn}(\lambda)\infty & \text{if } \gamma\in[1,\infty)\,.\end{cases}$

For the normal GARCH(1,1) $\tau_z=\lambda=0$ and thus $\mathrm{T}_{r,n}=\lim_{n\to\infty}\mathrm{T}_{r,n}=0$.

(c) Limits of forward and aggregated kurtosis

The forward kurtosis limit is:

$$\lim_{s\to\infty}\kappa_{r,s}=\kappa_z\lim_{s\to\infty}\left(\tilde{\mu}^{(1)}_{h,s}\right)^{-2}\lim_{s\to\infty}\tilde{\mu}^{(2)}_{h,s}=\begin{cases}\kappa_z\omega\bar{h}^{-2}\left(\omega+2\varphi\bar{h}\right)(1-\gamma)^{-1} & \text{if } \gamma\in(0,1)\,,\\ \infty & \text{if } \gamma\in[1,\infty)\,.\end{cases}$$

For the normal GJR and normal GARCH(1,1) the limit of the kurtosis of forward returns becomes:

$$\lim_{s\to\infty}\kappa_{r,s}=\begin{cases}3\omega\bar{h}^{-2}\left(\omega+2\varphi\bar{h}\right)(1-\gamma)^{-1} & \text{if } \gamma\in(0,1)\,,\\ \infty & \text{if } \gamma\in[1,\infty)\,,\end{cases}$$

where φ and γ are now given by (35) and (36) for the normal GJR and normal GARCH(1,1), respectively.

For the limit of the aggregated kurtosis, we write:

$$\lim_{n\to\infty}\mathrm{K}_{r,n}=\lim_{n\to\infty}\left(M^{(2)}_{r,n}\right)^{-2}M^{(4)}_{r,n}=\lim_{n\to\infty}\left(n^{-1}M^{(2)}_{r,n}\right)^{-2}\left(n^{-2}M^{(4)}_{r,n}\right)=\bar{h}^{-2}\lim_{n\to\infty}\left(n^{-2}M^{(4)}_{r,n}\right).$$

$$\lim_{n\to\infty}\left(n^{-2}M^{(4)}_{r,n}\right)=\kappa_zA_1+6A_2+4A_3, \tag{37}$$

where

$$A_1=\lim_{n\to\infty}\left(n^{-2}\sum_{s=1}^{n}\tilde{\mu}^{(2)}_{h,s}\right),\qquad A_2=\lim_{n\to\infty}\left(n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(\varepsilon^2_{t+s}\varepsilon^2_{t+s+u}\right)\right),$$

$$A_3=\lim_{n\to\infty}\left(n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(\varepsilon_{t+s}\varepsilon^3_{t+s+u}\right)\right)+3\lim_{n\to\infty}\left(n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon^2_{t+s+u+v}\right)\right).$$

Now, if $\gamma\ne1$,

$$\begin{aligned}A_1&=\lim_{n\to\infty}n^{-2}\left(nc_1+\left(\tilde{\mu}^{(2)}_{h,1}-c_3\right)(1-\gamma)^{-1}(1-\gamma^n)+c_2(1-\varphi)^{-1}(1-\varphi^n)\right)\\&=c_1\lim_{n\to\infty}n^{-1}+\left(h^2_{t+1}-c_3\right)(1-\gamma)^{-1}\lim_{n\to\infty}n^{-2}(1-\gamma^n)+c_2(1-\varphi)^{-1}\lim_{n\to\infty}n^{-2}(1-\varphi^n)\,.\end{aligned}$$

[25] If $c_{13}>0$, then $\tilde{S}_{1,n}\le S_n\le\tilde{S}_{2,n}$, whereas if $c_{13}<0$, the inequality is reversed. However, this does not change the proof above.

[26] It can be easily noticed that $\operatorname{sgn}(-c_{12}(\varphi-\gamma)+c_{13}(1-\gamma))=\operatorname{sgn}(c_{12})$, for $c_{14}<0$. Hence the limits of aggregated skewness are the same, regardless of c_{14} being greater than or less than 0.

Otherwise, when $\gamma = 1$:

$$\sum_{s=1}^{n} \tilde{\mu}_{h,s}^{(2)} = \sum_{s=1}^{n} \left((s-1)\left(\omega^2 + 2\omega\varphi\bar{h}\right) + 2\varphi h_0 \left(1 - \varphi^{s-1}\right)\left(h_{t+1} - \bar{h}\right) + h_{t+1}^2 \right)$$
$$= 1/2\left(\omega^2 + 2\omega\varphi\bar{h}\right) n^2 + \left(2\varphi\bar{h}\left(h_{t+1} - \bar{h}\right) - 1/2\left(\omega^2 + 2\omega\varphi\bar{h}\right) + h_{t+1}^2\right) n$$
$$-2\varphi\bar{h}\left(h_{t+1} - \bar{h}\right)(1-\varphi)^{-1}(1-\varphi^n).$$

So for $\gamma = 1$, we obtain that:

$$A_1 = \lim_{n\to\infty} 1/2\left(\omega^2 + 2\omega\varphi\bar{h}\right) + \lim_{n\to\infty} n^{-1}\left(2\varphi\bar{h}\left(h_{t+1} - \bar{h}\right) - 1/2\left(\omega^2 + 2\omega\varphi\bar{h}\right) + h_{t+1}^2\right)$$
$$- \lim_{n\to\infty} n^{-2} 2\varphi\bar{h}\left(h_{t+1} - \bar{h}\right)(1-\varphi)^{-1}(1-\varphi^n).$$

Hence

$$A_1 = \begin{cases} 0 & \text{if } \gamma \in (0,1), \\ 1/2\left(\omega^2 + 2\omega\varphi\bar{h}\right) & \text{if } \gamma = 1, \\ \infty & \text{if } \gamma \in (1,\infty). \end{cases} \tag{38}$$

For A_2, using the derivations from Appendix A.1, we can write:

$$A_2 = \lim_{n\to 0}\left(n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right)\right)$$
$$= \lim_{n\to 0}\left(n^{-2}\left[\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left[\bar{h}\left(1-\varphi^u\right)\tilde{\mu}_{h,s}^{(1)}\right] + \kappa_z\left(\alpha + \lambda F_0 + \kappa_z^{-1}\beta\right)\sum_{s=1}^{n}\sum_{u=1}^{n-2}\varphi^{u-1}\tilde{\mu}_{h,s}^{(2)}\right]\right).$$

For $\gamma \neq 1$, the expressions for the two double sums in the expressions above were derived in Appendix A.1. Using those results, we have:

$$A_2 = \lim_{n\to\infty} n^{-2}\left[\begin{array}{l} 1/2 n(n-1)\bar{h}^2 + (1-\varphi)^{-1}\bar{h}\left(h_{t+1} - \bar{h}\right) \\ \left(\left(n - (1-\varphi)^{-1}(1-\varphi^n)\right) - \varphi\left(\begin{array}{l} n\bar{h}\left(h_{t+1} - \bar{h}\right)^{-1} + (1-\varphi)^{-1}(1-\varphi^n) - h_0 \\ (h_{t+1} - h_0)^{-1}(1-\varphi)^{-1}(1-\varphi^n) - n\varphi^{n-1} \end{array}\right)\right) \\ +\kappa_z\left(\alpha + \lambda F_0 + \kappa_z{}^{-1}\beta\right)(1-\varphi)^{-1} \\ \left(\begin{array}{l} nc_1 + \left(\tilde{\mu}_{h,1}^{(2)} - c_3\right)(1-\gamma)^{-1}(1-\gamma^n) + c_2(1-\varphi)^{-1}(1-\varphi^n) \\ -c_1(1-\varphi)^{-1}(1-\varphi^n) - \left(h_{t+1}^2 - c_3\right)(\varphi-\gamma)^{-1}(\varphi^n - \gamma^n) - nc_2\varphi^{n-1} \end{array}\right) \end{array}\right]$$

For $\gamma = 1$,

$$A_2 = \lim_{n\to\infty} n^{-2} \left[\begin{array}{l} {}^1\!/\!_2 n\left(n-1\right)\bar{h}^2 + \left(1-\varphi\right)^{-1}\bar{h}\left(h_{t+1}-\bar{h}\right) \\ \left(\left(n - \left(1-\varphi\right)^{-1}\left(1-\varphi^n\right)\right) - \varphi \left(\begin{array}{l} n\bar{h}\left(h_{t+1}-\bar{h}\right)^{-1} + \left(1-\varphi\right)^{-1}\left(1-\varphi^n\right) - h_0 \\ \left(h_{t+1}-h_0\right)^{-1}\left(1-\varphi\right)^{-1}\left(1-\varphi^n\right) - n\varphi^{n-1} \end{array} \right) \right) \\ +\kappa_z\left(\alpha + \lambda F_0 + \kappa_z{}^{-1}\beta\right)\left(1-\varphi\right)^{-1} \\ \left(\begin{array}{l} {}^1\!/\!_2\left(\omega^2 + 2\omega\varphi\bar{h}\right)n^2 + \left(2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right) - {}^1\!/\!_2\left(\omega^2+2\omega\varphi\bar{h}\right) + h_{t+1}^2\right)n \\ -2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right)\left(1-\varphi\right)^{-1}\left(1-\varphi^n\right) \\ +\varphi\left(\omega^2+2\omega\varphi\bar{h}\right)\left(1-\varphi\right)^{-1}\left[\left(1-\varphi\right)^{-1}\left(1-\varphi^n\right) - \varphi^{-1}\left(n-1\right)\right] \\ -2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right)\left[\left(1-\varphi\right)^{-1}\left(1-\varphi^n\right) - n\varphi^{n-1}\right] - \tilde{\mu}_{h,1}^{(2)}\left(1-\varphi\right)^{-1}\left(1-\varphi^n\right). \end{array} \right) \end{array} \right]$$

Thus:

$$A_2 = \begin{cases} {}^1\!/\!_2\,\bar{h}^2 & \text{if } \gamma \in (0,1)\,, \\ {}^1\!/\!_2\left[\bar{h}^2 + \kappa_z\left(\alpha+\lambda F_0 + \kappa_z^{-1}\beta\right)\left(1-\varphi\right)^{-1}\left(\omega^2+2\omega\varphi\bar{h}\right)\right] & \text{if } \gamma = 1, \\ \infty & \text{if } \gamma \in (1,\infty)\,. \end{cases} \tag{39}$$

$$A_3 = \left[\tau_z + 3(1-\varphi)^{-1}c_9\right]\lim_{n\to\infty} n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\theta_{su}^{(3/2)} - 3(1-\varphi)^{-1}c_9\lim_{n\to\infty} n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\varphi^{n-s-u}\theta_{su}^{(3/2)}.$$

$$\begin{aligned} \theta_{su}^{(3/2)} &= \tfrac{3}{4}c_9\left[\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{1/2} + \omega\varphi(\varphi-\gamma)^{-1}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\right]\varphi^{u-1}E_t\left(h_{t+s}^{3/2}\right) \\ &\quad + \tfrac{3}{8}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\gamma^{u-1}\left(c_{10}E_t\left(h_{t+s}^{5/2}\right) + 2\omega\gamma(\gamma-\varphi)^{-1}c_9E_t\left(h_{t+s}^{3/2}\right)\right), \\ \varphi^{n-s-u}\theta_{su}^{(3/2)} &= \tfrac{3}{4}c_9\left[\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{1/2} + \omega\varphi(\varphi-\gamma)^{-1}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\right]\varphi^{n-s-1}E_t\left(h_{t+s}^{3/2}\right) \\ &\quad + \tfrac{3}{8}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\varphi^{n-s-1}(\gamma/\varphi)^{u-1}\left(c_{10}E_t\left(h_{t+s}^{5/2}\right) + 2\omega\gamma(\gamma-\varphi)^{-1}c_9E_t\left(h_{t+s}^{3/2}\right)\right). \end{aligned}$$

We can now write: $b_{l,s,n} \le \sum_{u=1}^{n-s}\theta_{su}^{(3/2)} \le b_{u,s,n}$, where, for $\gamma \neq 1$

$$\begin{aligned} b_{l,s,n} &= \tfrac{3}{4}\left(c_9\left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_9\right)\right]^{1/2} + \omega\varphi(\varphi-\gamma)^{-1}c_9\left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, (\varphi-\gamma)\,c_9\right)\right]^{-1/2}\right) \\ &\quad E_t\left(h_{t+s}^{3/2}\right)\left(1-\varphi\right)^{-1}\left(1-\varphi^{n-s}\right) \\ &\quad + \tfrac{3}{8}\left(\begin{array}{l} c_{10}\left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_{10}\right)\right]^{-1/2}E_t\left(h_{t+s}^{5/2}\right) \\ +2\omega\gamma(\gamma-\varphi)^{-1}c_9\left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, (\gamma-\varphi)\,c_9\right)\right]^{-1/2}E_t\left(h_{t+s}^{3/2}\right) \end{array} \right)\left(1-\gamma\right)^{-1}\left(1-\gamma^{n-s}\right), \end{aligned}$$

$$b_{u,s,n} = \frac{3}{4}\begin{pmatrix} c_9\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u}, c_9\right)\right]^{1/2} + \omega\varphi(\varphi-\gamma)^{-1} \\ c_9\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u}, -(\varphi-\gamma)\,c_9\right)\right]^{-1/2}\end{pmatrix} E_t\left(h^{3/2}_{t+s}\right)(1-\varphi)^{-1}\left(1-\varphi^{n-s}\right)$$
$$+\frac{3}{8}\begin{pmatrix} c_{10}\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u}, -c_{10}\right)\right]^{-1/2} E_t\left(h^{5/2}_{t+s}\right) + 2\omega\gamma(\gamma-\varphi)^{-1} \\ c_9\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u}, -(\gamma-\varphi)\,c_9\right)\right]^{-1/2} E_t\left(h^{3/2}_{t+s}\right)\end{pmatrix}(1-\gamma)^{-1}\left(1-\gamma^{n-s}\right),$$

and, for $\gamma = 1$

$$b_{l,s,n} = \frac{3}{4}\left(c_9\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u}, -c_9\right)\right]^{1/2} + \omega\varphi(\varphi-1)^{-1}c_9\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u}, -c_9\right)\right]^{-1/2}\right)$$
$$E_t\left(h^{3/2}_{t+s}\right)(1-\varphi)^{-1}\left(1-\varphi^{n-s}\right)$$
$$+\frac{3}{8}(n-s)\begin{pmatrix} c_{10}\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u}, c_{10}\right)\right]^{-1/2} E_t\left(h^{5/2}_{t+s}\right) \\ +2\omega(1-\varphi)^{-1}c_9\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u}, c_9\right)\right]^{-1/2} E_t\left(h^{3/2}_{t+s}\right)\end{pmatrix},$$

$$b_{u,s,n} = \frac{3}{4}\left(c_9\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u}, c_9\right)\right]^{1/2} + \omega\varphi(\varphi-1)^{-1}c_9\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u}, c_9\right)\right]^{-1/2}\right)$$
$$E_t\left(h^{3/2}_{t+s}\right)(1-\varphi)^{-1}\left(1-\varphi^{n-s}\right)$$
$$+\frac{3}{8}(n-s)\begin{pmatrix} c_{10}\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u}, -c_{10}\right)\right]^{-1/2} E_t\left(h^{5/2}_{t+s}\right) \\ +2\omega(1-\varphi)^{-1}c_9\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u}, -c_9\right)\right]^{-1/2} E_t\left(h^{3/2}_{t+s}\right)\end{pmatrix},$$

with $q_{z,a,b}(x_z, y) = \begin{cases} \max\limits_{a\le z\le b} x_z & \text{if } y \ge 0, \\ \min\limits_{a\le z\le b} x_z & \text{if } y < 0, \end{cases}$ where $z = u$ or $z = s$.

Also, $b_{ul,n} \le \sum\limits_{s=1}^{n} b_{u,s,n} \le b_{uu,n}$ where for $\gamma \ne 1$

$$b_{ul,n} = \frac{3}{4}(1-\varphi)^{-1}\sum_{s=1}^{n}\left(1-\varphi^{n-s}\right)$$
$$\begin{bmatrix} c_9 q_{s,1,n}\left(E_t\left(h^{3/2}_{t+s}\right), -c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u}, c_9\right), -c_9\right)\right]^{1/2} \\ +\omega\varphi(\varphi-\gamma)^{-1}c_9 q_{s,1,n}\left(E_t\left(h^{3/2}_{t+s}\right), -(\varphi-\gamma)\,c_9\right) \\ \left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u}, -(\varphi-\gamma)\,c_9\right), (\varphi-\gamma)\,c_9\right)\right]^{-1/2}\end{bmatrix}$$
$$+\frac{3}{8}(1-\gamma)^{-1}\sum_{s=1}^{n}\left(1-\gamma^{n-s}\right)$$
$$\begin{bmatrix} c_{10} q_{s,1,n}\left(E_t\left(h^{5/2}_{t+s}\right), -c_{10}\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u}, -c_{10}\right), c_{10}\right)\right]^{-1/2} \\ +2\omega\gamma(\gamma-\varphi)^{-1}c_9 q_{s,1,n}\left(E_t\left(h^{3/2}_{t+s}\right), -(\gamma-\varphi)\,c_9\right) \\ \left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u}, -(\gamma-\varphi)\,c_9\right), (\gamma-\varphi)\,c_9\right)\right]^{-1/2}\end{bmatrix}$$
$$= \frac{3}{4}l_{1,ul,n}(1-\varphi)^{-1}\left(n-(1-\varphi)^{-1}(1-\varphi^n)\right) + \frac{3}{8}l_{2,ul,n}(1-\gamma)^{-1}\left(n-(1-\gamma)^{-1}(1-\gamma^n)\right)$$

with

$$l_{1,ul,n} = c_9 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right),-c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)},c_9\right),-c_9\right)\right]^{1/2}$$
$$+\omega\varphi(\varphi-\gamma)^{-1}c_9 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right),-(\varphi-\gamma)c_9\right)$$
$$\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)},-(\varphi-\gamma)c_9\right),(\varphi-\gamma)c_9\right)\right]^{-1/2}$$

and

$$l_{2,ul,n} = c_{10} q_{s,1,n}\left(E_t\left(h_{t+s}^{5/2}\right),-c_{10}\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)},-c_{10}\right),c_{10}\right)\right]^{-1/2}$$
$$+2\omega\gamma(\gamma-\varphi)^{-1}c_9 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right),-(\gamma-\varphi)c_9\right)$$
$$\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)},-(\gamma-\varphi)c_9\right),(\gamma-\varphi)c_9\right)\right]^{-1/2}.$$

Also

$$b_{uu,n} = \tfrac{3}{4}(1-\varphi)^{-1}\sum_{s=1}^{n}\left(1-\varphi^{n-s}\right)$$
$$\left[\begin{array}{l} c_9 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right),c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)},c_9\right),c_9\right)\right]^{1/2}+\omega\varphi(\varphi-\gamma)^{-1}c_9 \\ q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right),(\varphi-\gamma)c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)},-(\varphi-\gamma)c_9\right),-(\varphi-\gamma)c_9\right)\right]^{-1/2}\end{array}\right]$$
$$+\tfrac{3}{8}(1-\gamma)^{-1}\sum_{s=1}^{n}\left(1-\gamma^{n-s}\right) \quad .$$
$$\left[\begin{array}{l} c_{10} q_{s,1,n}\left(E_t\left(h_{t+s}^{5/2}\right),c_{10}\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)},-c_{10}\right),-c_{10}\right)\right]^{-1/2}+2\omega\gamma(\gamma-\varphi)^{-1}c_9 \\ q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right),(\gamma-\varphi)c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)},-(\gamma-\varphi)c_9\right),-(\gamma-\varphi)c_9\right)\right]^{-1/2}\end{array}\right]$$
$$= \tfrac{3}{4}l_{1,uu,n}(1-\varphi)^{-1}\left(n-(1-\varphi)^{-1}(1-\varphi^n)\right)+\tfrac{3}{8}l_{2,uu,n}(1-\gamma)^{-1}\left(n-(1-\gamma)^{-1}(1-\gamma^n)\right)$$

with

$$l_{1,uu,n} = c_9 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right),c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)},c_9\right),c_9\right)\right]^{1/2}$$
$$+\omega\varphi(\varphi-\gamma)^{-1}c_9 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right),(\varphi-\gamma)c_9\right)$$
$$\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)},-(\varphi-\gamma)c_9\right),-(\varphi-\gamma)c_9\right)\right]^{-1/2}$$

and

$$l_{2,uu,n} = c_{10} q_{s,1,n}\left(E_t\left(h_{t+s}^{5/2}\right),c_{10}\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)},-c_{10}\right),-c_{10}\right)\right]^{-1/2}$$
$$+2\omega\gamma(\gamma-\varphi)^{-1}c_9 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right),(\gamma-\varphi)c_9\right) \quad .$$
$$\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)},-(\gamma-\varphi)c_9\right),-(\gamma-\varphi)c_9\right)\right]^{-1/2}.$$

For $\gamma = 1$:

$$b_{ul,n} = \tfrac{3}{4}\left[\begin{array}{l} c_9 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right),-c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)},c_9\right),-c_9\right)\right]^{1/2}+\omega\varphi(\varphi-1)^{-1}c_9 \\ q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right),c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)},c_9\right),-c_9\right)\right]^{-1/2}\end{array}\right]$$
$$(1-\varphi)^{-1}\sum_{s=1}^{n}\left(1-\varphi^{n-s}\right)+\tfrac{3}{16}n(n-1)$$
$$\left[\begin{array}{l} c_{10} q_{s,1,n}\left(E_t\left(h_{t+s}^{5/2}\right),-c_{10}\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)},-c_{10}\right),c_{10}\right)\right]^{-1/2} \\ +2\omega(1-\varphi)^{-1}c_9 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right),-c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)},-c_9\right),c_9\right)\right]^{-1/2}\end{array}\right]$$
$$= \tfrac{3}{4}\tilde{l}_{1,ul,n}(1-\varphi)^{-1}\left(n-(1-\varphi)^{-1}(1-\varphi^n)\right)+\tfrac{3}{16}\tilde{l}_{2,ul,n}n(n-1)$$

with

$$\tilde{l}_{1,ul,n} = c_9 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), -c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_9\right), -c_9\right)\right]^{1/2}$$
$$+\omega\varphi(\varphi-1)^{-1} c_9 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_9\right), -c_9\right)\right]^{-1/2}$$

and

$$\tilde{l}_{2,ul,n} = c_{10} q_{s,1,n}\left(E_t\left(h_{t+s}^{5/2}\right), -c_{10}\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_{10}\right), c_{10}\right)\right]^{-1/2}$$
$$+2\omega(1-\varphi)^{-1} c_9 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), -c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_9\right), c_9\right)\right]^{-1/2}.$$

Also,

$$b_{uu,n} = \tfrac{3}{4}(1-\varphi)^{-1}\sum_{s=1}^{n}\left(1-\varphi^{n-s}\right)$$
$$\left[\begin{array}{l} c_9 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_9\right), c_9\right)\right]^{1/2} + \omega\varphi(\varphi-1)^{-1} c_9 \\ q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), -c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_9\right), c_9\right)\right]^{-1/2} \end{array}\right]$$
$$+\tfrac{3}{16} n\,(n-1)$$
$$\left[\begin{array}{l} c_{10} q_{s,1,n}\left(E_t\left(h_{t+s}^{5/2}\right), c_{10}\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_{10}\right), -c_{10}\right)\right]^{-1/2} \\ +2\omega(1-\varphi)^{-1} c_9 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_9\right), -c_9\right)\right]^{-1/2} \end{array}\right]$$
$$= \tfrac{3}{4}\tilde{l}_{1,uu,n}(1-\varphi)^{-1}\left(n-(1-\varphi)^{-1}\left(1-\varphi^{n}\right)\right) + \tfrac{3}{16}\tilde{l}_{2,uu,n}\, n\,(n-1).$$

where

$$\tilde{l}_{1,uu,n} = c_9 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_9\right), c_9\right)\right]^{1/2}$$
$$+\omega\varphi(\varphi-1)^{-1} c_9 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), -c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_9\right), c_9\right)\right]^{-1/2}$$

and

$$\tilde{l}_{2,uu,n} = c_{10} q_{s,1,n}\left(E_t\left(h_{t+s}^{5/2}\right), c_{10}\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_{10}\right), -c_{10}\right)\right]^{-1/2}$$
$$+2\omega(1-\varphi)^{-1} c_9 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), c_9\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_9\right), -c_9\right)\right]^{-1/2}.$$

We have previously shown that $L_{\max} = \lim_{n\to\infty}\left(\max_{1\le s\le n}\ \tilde{\mu}_{h,s}^{(1)}\right) = \max\left(\bar{h}, h_{t+1}\right)$ and $L_{\min} = \lim_{n\to\infty}\left(\min_{1\le s\le n}\ \tilde{\mu}_{h,s}^{(1)}\right) = \min\left(\bar{h}, h_{t+1}\right)$. Also, we have:

$$\lim_{s\to\infty} E_t\left(h_{t+s}^{3/2}\right) = \tfrac{1}{8}\lim_{s\to\infty}\left(5\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3/2} + 3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-1/2}\right)$$
$$= \begin{cases} \tfrac{1}{8}\left(5\bar{h}^{3/2} + 3\bar{h}^{-1/2}\left(c_1 + \left(h_{t+1}^2 - c_3\right)\lim_{s\to\infty}\gamma^{s-1} + c_2\lim_{s\to\infty}\varphi^{s-1}\right)\right) & \text{if } \gamma\neq 1 \\ \tfrac{1}{8}\left(5\bar{h}^{3/2} + 3\bar{h}^{-1/2}\lim_{s\to\infty}\left(\begin{array}{l}(s-1)\left(\omega^2+2\omega\varphi\bar{h}\right) \\ +2\varphi\bar{h}\left(1-\varphi^{s-1}\right)\left(h_{t+1}-\bar{h}\right) + h_{t+1}^2\end{array}\right)\right) & \text{if } \gamma = 1 \end{cases}$$
$$= \begin{cases} \tfrac{1}{8}\left(5\bar{h}^{3/2} + 3\bar{h}^{-1/2}c_1\right) & \text{if } \gamma\in(0,1) \\ \infty & \text{if } \gamma\in[1,\infty). \end{cases}$$

$$\lim_{s\to\infty} E_t\left(h_{t+s}^{5/2}\right) = \frac{1}{8}\lim_{s\to\infty}\left(\left(15\left(\tilde{\mu}_{h,s}^{(1)}\right)^{1/2}\tilde{\mu}_{h,s}^{(2)} - 7\left(\tilde{\mu}_{h,s}^{(1)}\right)^{5/2}\right)\right) = \begin{cases} \frac{1}{8}\left(-7\bar{h}^{5/2} + 15\bar{h}^{1/2}c_1\right) & \text{if } \gamma\in(0,1) \\ \infty & \text{if } \gamma\in[1,\infty) \end{cases}.$$

Hence, when $\gamma \in (0,1)$ we obtained that $\lim\limits_{s\to\infty} E_t\left(h_{t+s}^{3/2}\right)$ and $\lim\limits_{s\to\infty} E_t\left(h_{t+s}^{5/2}\right)$ exist and are finite. Furthermore, we get that $\max\limits_{1\le s\le n} E_t\left(h_{t+s}^{i/2}\right)$ and $\min\limits_{1\le s\le n} E_t\left(h_{t+s}^{i/2}\right)$, $i = 3$ and 5 are bounded. Thus, $\lim\limits_{n\to\infty}\left(l_{j,uu,n}\right)$ and $\lim\limits_{n\to\infty}\left(l_{j,ul,n}\right)$, $j=1$ and 2 exist and are finite. We get:

$$\lim_{n\to\infty} n^{-2} b_{ul,n} = \lim_{n\to\infty} n^{-2}\begin{pmatrix} \frac{3}{4} l_{1,ul,n}(1-\varphi)^{-1}\left(n-(1-\varphi)^{-1}(1-\varphi^n)\right) \\ +\frac{3}{8} l_{2,ul,n}(1-\gamma)^{-1}\left(n-(1-\gamma)^{-1}(1-\gamma^n)\right)\end{pmatrix} = 0$$

$$\lim_{n\to\infty} n^{-2} b_{uu,n} = \lim_{n\to\infty} n^{-2}\begin{bmatrix} \frac{3}{4} l_{1,uu,n}(1-\varphi)^{-1}\left(n-(1-\varphi)^{-1}(1-\varphi^n)\right) \\ +\frac{3}{8} l_{2,uu,n}(1-\gamma)^{-1}\left(n-(1-\gamma)^{-1}(1-\gamma^n)\right)\end{bmatrix} = 0$$

Thus $\lim\limits_{n\to\infty} n^{-2}\sum\limits_{s=1}^{n} b_{u,s,n} = 0$. This translates into: $\lim\limits_{n\to\infty} n^{-2}\sum\limits_{s=1}^{n}\sum\limits_{u=1}^{n-s}\theta_{su}^{(3/2)} \le \lim\limits_{n\to\infty} n^{-2}\sum\limits_{s=1}^{n} b_{u,s,n} = 0$. Similarly, it can be shown that: $0 = \lim\limits_{n\to\infty} n^{-2}\sum\limits_{s=1}^{n} b_{l,s,n} \le \lim\limits_{n\to\infty} n^{-2}\sum\limits_{s=1}^{n}\sum\limits_{u=1}^{n-s}\theta_{su}^{(3/2)}$. Finally, we obtain that $\lim\limits_{n\to\infty} n^{-2}\sum\limits_{s=1}^{n}\sum\limits_{u=1}^{n-s}\theta_{su}^{(3/2)} = 0$ for $\gamma\in(0,1)$. Similarly it can be shown that $\lim\limits_{n\to\infty} n^{-2}\sum\limits_{s=1}^{n}\sum\limits_{u=1}^{n-s}\varphi^{n-s-u}\theta_{su}^{(3/2)} = 0$ for $\gamma\in(0,1)$. Consider now: $B_{l,s,n} \le \sum\limits_{u=1}^{n-s}\varphi^{n-s-u}\theta_{su}^{3/2} \le B_{u,s,n}$ where, for any γ:

$$\begin{aligned} B_{l,s,n} = {} & \frac{3}{4}\left(c_9\left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_9\right)\right]^{1/2} + \omega\varphi(\varphi-\gamma)^{-1}c_9\left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, (\varphi-\gamma)c_9\right)\right]^{-1/2}\right) \\ & E_t\left(h_{t+s}^{3/2}\right)(n-s)\varphi^{n-s-1} \\ & +\frac{3}{8}\begin{pmatrix} c_{10}\left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_{10}\right)\right]^{-1/2} E_t\left(h_{t+s}^{5/2}\right) + 2\omega\gamma(\gamma-\varphi)^{-1}c_9 \\ \left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, (\gamma-\varphi)c_9\right)\right]^{-1/2} E_t\left(h_{t+s}^{3/2}\right)\end{pmatrix}(\varphi-\gamma)^{-1}\left(\varphi^{n-s}-\gamma^{n-s}\right), \\ B_{u,s,n} = {} & \frac{3}{4}\left(c_9\left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_9\right)\right]^{1/2} + \omega\varphi(\varphi-\gamma)^{-1}c_9\left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -(\varphi-\gamma)c_9\right)\right]^{-1/2}\right) \\ & E_t\left(h_{t+s}^{3/2}\right)(n-s)\varphi^{n-s-1} \\ & +\frac{3}{8}\begin{pmatrix} c_{10}\left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_{10}\right)\right]^{-1/2} E_t\left(h_{t+s}^{5/2}\right) + 2\omega\gamma(\gamma-\varphi)^{-1}c_9 \\ \left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -(\gamma-\varphi)c_9\right)\right]^{-1/2} E_t\left(h_{t+s}^{3/2}\right)\end{pmatrix}(\varphi-\gamma)^{-1}\left(\varphi^{n-s}-\gamma^{n-s}\right). \end{aligned}$$

Also, $B_{ul,n} \le \sum\limits_{s=1}^{n} B_{u,s,n} \le B_{uu,n}$ where for $\gamma \ne 1$:

$$\begin{aligned} B_{ul,n} = {} & \frac{3}{4} l_{1,ul,n}\left[(1-\varphi)^{-2}(1-\varphi^n) - (1-\varphi)^{-1}n\varphi^{n-1}\right] \\ & +\frac{3}{8} l_{2,ul,n}\left[(\varphi-\gamma)^{-1}\left((1-\varphi)^{-1}(1-\varphi^n) - (1-\gamma)^{-1}(1-\gamma^n)\right)\right] \end{aligned}$$

and

$$B_{uu,n} = \tfrac{3}{4} l_{1,uu,n} \left[(1-\varphi)^{-2} (1-\varphi^n) - (1-\varphi)^{-1} n\varphi^{n-1} \right] + \tfrac{3}{8} l_{2,uu,n} \left[(\varphi-\gamma)^{-1} \left((1-\varphi)^{-1} (1-\varphi^n) - (1-\gamma)^{-1} (1-\gamma^n) \right) \right].$$

For $\gamma = 1$:

$$B_{ul,n} = \tfrac{3}{4} \tilde{l}_{1,ul,n} \left[(1-\varphi)^{-2} (1-\varphi^n) - (1-\varphi)^{-1} n\varphi^{n-1} \right] + \tfrac{3}{8} \tilde{l}_{2,ul,n} (1-\varphi)^{-1} \left(n - (1-\varphi)^{-1} (1-\varphi^n) \right)$$

and

$$B_{uu,n} = \tfrac{3}{4} \tilde{l}_{1,uu,n} \left[(1-\varphi)^{-2} (1-\varphi^n) - (1-\varphi)^{-1} n\varphi^{n-1} \right] + \tfrac{3}{8} \tilde{l}_{2,uu,n} (1-\varphi)^{-1} \left(n - (1-\varphi)^{-1} (1-\varphi^n) \right).$$

For $\gamma \in (0,1)$ we obtained that $\lim_{n\to\infty} (l_{j,uu,n})$, $\lim_{n\to\infty} (l_{j,ul,n})$, $j = 1, 2$ exist and are finite. Thus:

$$\lim_{n\to\infty} n^{-2} B_{ul,n} = \lim_{n\to\infty} n^{-2} \begin{bmatrix} \tfrac{3}{4} l_{1,ul,n} \left[(1-\varphi)^{-2} (1-\varphi^n) - (1-\varphi)^{-1} n\varphi^{n-1} \right] + \tfrac{3}{8} l_{2,ul,n} \\ \left[(\varphi-\gamma)^{-1} \left((1-\varphi)^{-1} (1-\varphi^n) - (1-\gamma)^{-1} (1-\gamma^n) \right) \right] \end{bmatrix} = 0$$

and

$$\lim_{n\to\infty} n^{-2} B_{uu,n} = \lim_{n\to\infty} n^{-2} \begin{bmatrix} \tfrac{3}{4} l_{1,uu,n} \left[(1-\varphi)^{-2} (1-\varphi^n) - (1-\varphi)^{-1} n\varphi^{n-1} \right] + \tfrac{3}{8} l_{2,uu,n} \\ \left[(\varphi-\gamma)^{-1} \left((1-\varphi)^{-1} (1-\varphi^n) - (1-\gamma)^{-1} (1-\gamma^n) \right) \right] \end{bmatrix} = 0$$

Thus $\lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} B_{u,s,n} = 0$, yielding: $\lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} \sum_{u=1}^{n-s} \varphi^{n-s-u} \theta_{su}^{(3/2)} \leq \lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} B_{u,s,n} = 0$. Similarly, it can be shown that: $0 = \lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} B_{l,s,n} \leq \lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} \sum_{u=1}^{n-s} \varphi^{n-s-u} \theta_{su}^{(3/2)}$. Finally, we get: $\lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} \sum_{u=1}^{n-s} \varphi^{n-s-u} \theta_{su}^{(3/2)} = 0$; we also showed above that $\lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} \sum_{u=1}^{n-s} \theta_{su}^{(3/2)} = 0$, therefore, $A_3 = 0$ for $\gamma \in (0,1)$.

For $\gamma \in (1,\infty)$, we showed that $A_1 = A_2 = \infty$. Thus if we can show that A_3 is bounded below, then the kurtosis of aggregated returns will diverge to plus infinity in this case. Setting $c_{17} = \tau_z + 3(1-\varphi)^{-1} c_9$, we can then write:

$$A_3 = \lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} \sum_{u=1}^{n-s} \begin{bmatrix} \tfrac{3}{4} \left(\tilde{\mu}_{h,s+u}^{(1)} \right)^{-1/2} c_9 \left(c_{17} + (\tau_z - c_{17}) \varphi^{n-s-u} \right) \\ \left[\tilde{\mu}_{h,s+u}^{(1)} \varphi^{u-1} + \omega(\varphi-\gamma)^{-1} (\varphi^u - \gamma^u) \right] E_t \left(h_{t+s}^{3/2} \right) \\ + \tfrac{3}{8} \left(\tilde{\mu}_{h,s+u}^{(1)} \right)^{-1/2} \gamma^{u-1} c_{10} \left(c_{17} + (\tau_z - c_{17}) \varphi^{n-s-u} \right) E_t \left(h_{t+s}^{5/2} \right) \end{bmatrix}$$

It is reasonable to assume that $\mathrm{sgn}(\tau_z) = -\mathrm{sgn}(\lambda)$ when $\tau_z \neq 0$, since a positive λ means that volatility is more responsive to negative shocks rather than positive shocks of the same magnitude and translates into a negative skew of the aggregated returns. If $\mathrm{sgn}(\tau_z) = -\mathrm{sgn}(\lambda) \neq 0$, then $\mathrm{sgn}(\tau_z) = \mathrm{sgn}(c_9) = \mathrm{sgn}(c_{17})$, and consequently $\mathrm{sgn}\left(c_9 \left(c_{17} + (\tau_z - c_{17}) \varphi^{n-s-u}\right)\right) = \mathrm{sgn}\left(c_9 \left(\tau_z + 3(1-\varphi)^{-1} c_9 \left(1 - \varphi^{n-s-u}\right)\right)\right) = 1$. For $\mathrm{sgn}(\tau_z) = \mathrm{sgn}\left(\mu_z^{(5)}\right)$,[27] $\mathrm{sgn}(c_9) = \mathrm{sgn}(c_{10})$

[27] This is a sufficient but not necessary condition.

and $\operatorname{sgn}\left(c_{10}\left[c_{17}+\varphi^{n-s-u}\left(\tau_z-c_{17}\right)\right]\right) = \operatorname{sgn}\left(c_{10}\left(\tau_z+3(1-\varphi)^{-1}c_9\left(1-\varphi^{n-s-u}\right)\right)\right) = 1$. Hence all terms in A_3 are positive. We have thus shown that A_3 is bounded below (by 0), and hence the limit $\lim_{n\to\infty} \mathrm{K}_{r,n} = \infty$ for $\gamma \in (1,\infty)$.

For $\gamma = 1$, we can write:

$$\theta_{su}^{(3/2)} = \tfrac{3}{4}c_9\left[\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{1/2} + \omega\varphi(\varphi-1)^{-1}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\right]\varphi^{u-1}E_t\left(h_{t+s}^{3/2}\right)$$
$$+\tfrac{3}{8}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\left(c_{10}E_t\left(h_{t+s}^{5/2}\right)+2\omega(1-\varphi)^{-1}c_9E_t\left(h_{t+s}^{3/2}\right)\right)$$

and

$$\varphi^{n-s-u}\theta_{su}^{(3/2)} = \tfrac{3}{4}c_9\left[\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{1/2} + \omega\varphi(\varphi-1)^{-1}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\right]\varphi^{n-s-1}E_t\left(h_{t+s}^{3/2}\right)$$
$$+\tfrac{3}{8}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\varphi^{n-s-u}\left(c_{10}E_t\left(h_{t+s}^{5/2}\right)+2\omega(1-\varphi)^{-1}c_9E_t\left(h_{t+s}^{3/2}\right)\right)$$

Thus, for $\gamma = 1$, the expression for A_3 is:

$$A_3 = \lim_{n\to\infty} n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\begin{array}{l}\tfrac{3}{4}c_9\left(c_{17}+\left(\tau_z-c_{17}\right)\varphi^{n-s-u}\right)\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\\ \left[\tilde{\mu}_{h,s+u}^{(1)}\varphi^{u-1}+\omega(1-\varphi)^{-1}\left(1-\varphi^{u}\right)\right]E_t\left(h_{t+s}^{3/2}\right)\\ +\tfrac{3}{8}c_{10}\left(c_{17}+\left(\tau_z-c_{17}\right)\varphi^{n-s-u}\right)\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}E_t\left(h_{t+s}^{5/2}\right)\end{array}\right)$$

Since c_9, c_{10}, and c_{17} do not depend on γ, we still have that $\operatorname{sgn}(c_9) = \operatorname{sgn}(c_{10}) = \operatorname{sgn}(c_{17})$ and that: $\operatorname{sgn}\left(c_9\left(c_{17}+\left(\tau_z-c_{17}\right)\varphi^{n-s-u}\right)\right) = \operatorname{sgn}\left(c_{10}\left(c_{17}+\left(\tau_z-c_{17}\right)\varphi^{n-s-u}\right)\right) = 1$. Now we write:

$$A_3 \geq \lim_{n\to\infty} n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tfrac{3}{8}c_{10}\left(c_{17}+\left(\tau_z-c_{17}\right)\varphi^{n-s-u}\right)\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}E_t\left(h_{t+s}^{5/2}\right) \geq$$
$$\tfrac{3}{8}\lim_{n\to\infty}\left[\max_{1\leq s\leq n}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)\right]^{-1/2}\lim_{n\to\infty} n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}c_{10}\left(c_{17}+\left(\tau_z-c_{17}\right)\varphi^{n-s-u}\right)E_t\left(h_{t+s}^{5/2}\right) \geq$$
$$\tfrac{3}{64}\lim_{n\to\infty}\left[\max_{1\leq s\leq n}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)\right]^{-1/2}\lim_{n\to\infty}\left[\min_{1\leq s\leq n}\left[\min_{1\leq u\leq n-s}\left[c_{10}\left(c_{17}+\left(\tau_z-c_{17}\right)\varphi^{n-s-u}\right)\right]\right]\right]$$
$$\lim_{n\to\infty} n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\left(15\left(\tilde{\mu}_{h,s}^{(1)}\right)^{1/2}\tilde{\mu}_{h,s}^{(2)}-7\left(\tilde{\mu}_{h,s}^{(1)}\right)^{5/2}\right)\right)$$

Since $\varphi \in (0,1)$, both $\lim_{n\to\infty}\left[\min_{1\leq s\leq n}\left[\min_{1\leq u\leq n-s}\left[c_{10}\left(c_{17}+\left(\tau_z-c_{17}\right)\varphi^{n-s-u}\right)\right]\right]\right]$ and $\lim_{n\to\infty}\left[\max_{1\leq s\leq n}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)\right]$ are finite and we have shown above that they are positive, if $\tau_z \neq 0$ or $\lambda \neq 0$. In this case,

$$\lim_{n\to\infty} n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\left(15\left(\tilde{\mu}_{h,s}^{(1)}\right)^{1/2}\tilde{\mu}_{h,s}^{(2)}-7\left(\tilde{\mu}_{h,s}^{(1)}\right)^{5/2}\right)\right)$$
$$= \lim_{n\to\infty} n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\left(15\left(\tilde{\mu}_{h,s}^{(1)}\right)^{1/2}\left(\begin{array}{l}(s-1)\left(\omega^2+2\omega\varphi\bar{h}\right)\\ +2\varphi\bar{h}\left(1-\varphi^{s-1}\right)\left(h_{t+1}-\bar{h}\right)+h_{t+1}^2\end{array}\right)-7\left(\tilde{\mu}_{h,s}^{(1)}\right)^{5/2}\right)\right)$$
$$= 15\left(\omega^2+2\omega\varphi\bar{h}\right)\lim_{n\to\infty} n^{-2}\sum_{s=1}^{n}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{1/2}(n-s)(s-1)$$
$$\geq 15\left(\omega^2+2\omega\varphi\bar{h}\right)\lim_{n\to\infty}\left[\min_{1\leq s\leq n}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)\right]\lim_{n\to\infty} n^{-2}\left(\tfrac{n^2(n-5)}{6}\right),$$

where $15\left(\omega^2+2\omega\varphi\bar{h}\right)\lim_{n\to\infty}\left[\min_{1\leq s\leq n}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)\right]$ is positive and finite and consequently $A_3 = \infty$ for $\gamma = 1$ and $\tau_z \neq 0$ or $\lambda \neq 0$, while for $\gamma = 1$ and $\tau_z = \gamma = 0$, $c_9 = c_{10} = 0$ and $A_3 = 0$. Finally, we get that:

$$A_3 = \begin{cases} 0 & \text{if } \gamma \in (0,1) \\ \operatorname{sgn}(|\lambda| + |\tau_z|)\,\infty & \text{if } \gamma = 1 \\ \infty & \text{if } \gamma \in (1,\infty)\,, \end{cases}$$

which yields the final expression for the limit of aggregated kurtosis given in (10).

For the normal GJR, $\tau_z = 0$, but the convergence pattern of the aggregated kurtosis does not change and the limit of aggregated kurtosis is still infinite for $\gamma = 1$, as in the generic case. For the normal GARCH(1,1), $\tau_z = \lambda = 0$ and the limit of the aggregated kurtosis is now finite for $\gamma = 1$, although different from the normal value of 3. Thus, for the normal GARCH(1,1), we get:

$$\lim_{n\to\infty} \mathrm{K}_{r,n} = \begin{cases} 3 & \text{if } \gamma \in (0,1) \\ 3\left[1 + \frac{1}{2}(1+\alpha+\beta)(1+5\alpha+\beta)\right] & \text{if } \gamma = 1 \\ \infty & \text{if } \gamma \in (1,\infty)\ . \end{cases}$$

A.4: Limits of the Moments of Variances

This appendix derives the limits of the moments of forward and aggregated variances of the generic GJR model as the time horizon increases. In what follows we use the notation and assumptions defined at the start of the Appendix; additionally, we assume $c_4 \neq \varphi$, $c_4 \neq \gamma$.

(a) Limit of the Variance of Forward Variance

$$\lim_{s\to\infty} \mu^{(2)}_{h,s} = \lim_{s\to\infty} \left(\tilde{\mu}^{(2)}_{h,s} - \left(\tilde{\mu}^{(1)}_{h,s}\right)^2\right)$$

$$= \begin{cases} \lim\limits_{s\to\infty} \left[\begin{array}{l} \left(c_1 - \bar{h}^2\right) + \left(-c_3 + h^2_{t+1}\right)\gamma^{s-1} \\ + \left[c_2 - 2\bar{h}\left(h_{t+1} - \bar{h}\right)\right]\varphi^{s-1} - \varphi^{2(s-1)}\left(h_{t+1} - \bar{h}\right)^2 \end{array}\right] & \text{if } \gamma \neq 1 \\ \lim\limits_{s\to\infty} \left[\begin{array}{l} \left(2\omega\varphi\bar{h}\left(h_{t+1} - \bar{h}\right) + h^2_{t+1} - \bar{h}^2\right) + (s-1)\left(\omega^2 + 2\omega\varphi\bar{h}\right) \\ -2\bar{h}\left(\omega\varphi - 1\right)\left(h_{t+1} - \bar{h}\right)\varphi^{s-1} - \varphi^{2(s-1)}\left(h_{t+1} - \bar{h}\right)^2 \end{array}\right] & \text{if } \gamma = 1 \end{cases}$$

$$= \begin{cases} \left(c_1 - \bar{h}^2\right) & \text{if } \gamma \in (0,1) \\ \infty & \text{if } \gamma \in [1,\infty) \end{cases}$$

where $c_1 - \bar{h}^2 > 0$ and $h^2_{t+1} - c_3 > 0$.

(b)Limit of the Variance of Aggregated Variance

For $\gamma \neq 1$, using (24) and (28) - (30), the expression for the variance of the aggregated variance can be written as:[28] $M_{h,n}^{(2)} = An + B\gamma^n + C_n$ where $A = \left(c_1 - \bar{h}^2\right)\left(1 + 2\varphi(1-\varphi)^{-1}\right)$,
$B = -\left(\tilde{\mu}_{h,1}^{(2)} - c_3\right)(1-\gamma)^{-1} - 2\varphi(1-\varphi)^{-1}\left[\left(\tilde{\mu}_{h,1}^{(2)} - c_3\right)(1-\gamma)^{-1} - (\varphi-\gamma)^{-1}\right]$, $C_n = f\left(n\varphi^n, \varphi^n\right)$.
Since $\varphi \in (0,1)$, $\lim_{n\to\infty} \varphi^n = \lim_{n\to\infty} n\varphi^n = 0$ and $\exists\, C$ finite such that[29]: $\lim_{n\to\infty} C_n = \mathrm{C}$. Thus, the limit of the conditional variance of the aggregated conditional variance becomes:

$$\lim_{n\to\infty} M_{h,n}^{(2)} = \begin{cases} \operatorname{sgn}(A)\,\infty, & \text{if } \gamma \in (0,1) \\ \operatorname{sgn}(B)\,\infty & \text{if } \gamma \in (1,\infty), \end{cases} \tag{40}$$

where it can be easily seen that $\operatorname{sgn}(A) = 1$ and $\operatorname{sgn}(B) = 1$.

For $\gamma = 1$, using (22) - (23), (27), and (29) - (33), the expression for the variance of the aggregated variance becomes: $M_{h,n}^{(2)} = A'n^2 + B'n + C'$ where[30] $A' = \frac{1}{2}\left[2\varphi(1-\varphi)^{-1} + 1\right]\left(\omega^2 + 2\omega\varphi\bar{h}\right) > 0$. Hence: $\lim_{n\to\infty} M_{h,n}^{(2)} = \infty$ for any γ.

As with the variance of aggregated returns, the conditional variance of the aggregated conditional variance diverges to infinity when we increase the time horizon infinitely. It is meaningful to compute the limit of the daily variance, i.e. the variance divided by time. However, unlike the daily (one-period) variance of aggregated returns which converges to the level of (daily) unconditional variance, the daily conditional variance of aggregated conditional variance diverges to infinity under certain parameter conditions:[31]

$$\lim_{n\to\infty} \frac{M_{h,n}^{(2)}}{n} = \begin{cases} \lim_{n\to\infty}\left(A + B\frac{\gamma^n}{n} + \frac{C_n}{n}\right) & \text{if } \gamma \neq 1 \\ \lim_{n\to\infty}\left(A'n + B' + \frac{C'}{n}\right) & \text{if } \gamma = 1 \end{cases} = \begin{cases} A & \text{if } \gamma \in (0,1) \\ \infty & \text{if } \gamma \in [1,\infty) \end{cases}$$

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[28]The n^2 terms cancel out.

[29]Showing that $\lim_{n\to\infty} n\varphi^n = 0$ for $\varphi \in (0,1)$ is rather immediate. If we define $y = 1/\varphi$, then y > 1. We now have $\lim_{n\to\infty} n\varphi^n = \lim_{n\to\infty} \frac{n}{y^n} = 0$.

[30]As in the $\gamma \neq 1$ case above, $C' = f\left(n\varphi^n, \varphi^n\right)$ and $\lim_{n\to\infty} C' = \text{constant}$. B' is a constant, but its value and sign are irrelevant for the limit above.

[31]Unlike the conditional variance of the aggregated returns, which only depended on (powers of) the φ parameter which only take values between 0 and 1, the conditional variance of the aggregated variance also depends on (powers of) the γ parameter, which can take any positive value.

(c) Limit of the Skewness of Forward Variance

$$\lim_{s\to\infty} \tau_{h,s} = \lim_{s\to\infty} \left(\mu_{h,s}^{(3)} \left(\mu_{h,s}^{(2)} \right)^{-3/2} \right)$$

For $\gamma \in (0,1)$, we can write: $\lim_{s\to\infty} \tau_{h,s} = \frac{\lim_{s\to\infty}\left(\mu_{h,s}^{(3)}\right)}{\lim_{s\to\infty}\left(\mu_{h,s}^{(2)}\right)^{3/2}}$, where $\lim_{s\to\infty} \mu_{h,s}^{(2)} = c_1 - \bar{h}^2$. Using (34), we get:

$$\lim_{s\to\infty} \tau_{h,s} = \begin{cases} M_1 & \text{if } \gamma \in (0,1) \text{ and } c_4 \in (0,1) \\ \infty & \text{if } \gamma \in (0,1) \text{ and } c_4 \in [1,\infty) \ . \end{cases} \tag{41}$$

where $M_1 = \frac{\omega\left(\omega^2+3\omega\varphi\bar{h}+3\gamma c_1\right)(1-c_4)^{-1}-3\bar{h}c_1+2\bar{h}^3}{\left(c_1-\bar{h}^2\right)^{3/2}}$.

For $\gamma \in (1,\infty)$, we can write: $\lim_{s\to\infty} \tau_{h,s} = \frac{\lim_{s\to\infty}\left(\gamma^{-(3/2)s}\mu_{h,s}^{(3)}\right)}{\left[\lim_{s\to\infty}\left(\gamma^{-s}\mu_{h,s}^{(2)}\right)\right]^{3/2}}$ where, $\lim_{s\to\infty}\left(\gamma^{-s}\mu_{h,s}^{(2)}\right) = \left(-c_3 + h_{t+1}^2\right)$. Using (34), we get:

$$\lim_{s\to\infty} \tau_{h,s} = \begin{cases} \infty & \text{if } \gamma \in (1,\infty) \text{ and } c_4 > \gamma^{3/2} \\ 0 & \text{if } \gamma \in (1,\infty) \text{ and } c_4 \in \left(0,\gamma^{3/2}\right) \\ M_2 & \text{if } \gamma \in (1,\infty) \text{ and } c_4 = \gamma^{3/2} \ . \end{cases} \tag{42}$$

where $M_2 = \frac{c_{18}\gamma^{3/2}}{c_4\left(-c_3+h_{t+1}^2\right)^{3/2}} = \gamma^{3/2}\frac{h_{t+1}^3-\omega\left(\omega^2+3\omega\varphi\bar{h}+3\gamma c_1\right)(1-c_4)^{-1}-\left(3\omega^2\varphi\left(h_{t+1}-\bar{h}\right)+3\omega\gamma\left(-c_1+h_{t+1}^2\right)\right)(\varphi-c_4)^{-1}}{c_4\left(-c_3+h_{t+1}^2\right)^{3/2}}$.

For $\gamma = 1$, we can write: $\lim_{s\to\infty} \tau_{h,s} = \frac{\lim_{s\to\infty}\left(s^{-3/2}\mu_{h,s}^{(3)}\right)}{\left[\lim_{s\to\infty}\left(s^{-1}\mu_{h,s}^{(2)}\right)\right]^{3/2}}$, where, $\lim_{s\to\infty}\left(s^{-1}\mu_{h,s}^{(2)}\right) = \left(\omega^2 + 2\omega\varphi\bar{h}\right)$. In this case, the limit becomes:

$$\lim_{s\to\infty} \tau_{h,s} = \lim_{s\to\infty}\left(s^{-3/2}\tilde{\mu}_{h,s}^{(3)}\right) = \begin{cases} 0 & \text{if } c_4 \in (0,1) \\ \infty & \text{if } c_4 \in (1,\infty) \ . \end{cases} \tag{43}$$

Now, (41) - (43) give (13) in Theorem 4.

(d) Limit of the Skewness of Aggregated Variance

$$\lim_{n\to\infty} \mathrm{T}_{h,n} = \lim_{n\to\infty}\left[M_{h,n}^{(3)}\left(M_{h,n}^{(2)}\right)^{-3/2}\right]$$

For $\gamma \in (0,1)$, we can write: $\lim_{n\to\infty} \mathrm{T}_{h,n} = \frac{\lim_{n\to\infty}\left(n^{-3/2}M_{h,n}^{(3)}\right)}{\left[\lim_{n\to\infty}\left(n^{-1}M_{h,n}^{(2)}\right)\right]^{3/2}}$,

where $\lim_{n\to\infty}\left(n^{-1}M_{h,n}^{(2)}\right) = \left(c_1 - \bar{h}^2\right)\left(1 + 2\varphi(1-\varphi)^{-1}\right)$. Also, we write:

$$n^{-3/2}M_{h,n}^{(3)} = L_1 - 3L_2 + 2L_3 + 3\left(L_4 + L_5 + 2\left(L_6 + L_7 - L_8 - L_9\right) - L_{10} - L_{11}\right) \\ +6\left(L_{12} - L_{13} - L_{14} - L_{15} + 2L_{16}\right),$$

where

$$L_1 = \frac{\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(3)}}{n^{3/2}}, L_2 = \frac{\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)}\tilde{\mu}_{h,s}^{(1)}}{n^{3/2}}, L_3 = \frac{\sum_{s=1}^{n}\left(\tilde{\mu}_{h,s}^{(1)}\right)^3}{n^{3/2}} L_4 = \frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}_{h,su}^{(2,1)}}{n^{3/2}}, L_5 = \frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}_{h,su}^{(1,2)}}{n^{3/2}},$$

$$L_6 = \frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\tilde{\mu}_{h,s}^{(1)}\right)^2\tilde{\mu}_{h,s+u}^{(1)}}{n^{3/2}}, L_7 = \frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^2\tilde{\mu}_{h,s}^{(1)}}{n^{3/2}}, L_8 = \frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,su}^{(1,1)}}{n^{3/2}}, L_9 = \frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}_{h,s+u}^{(1)}\tilde{\mu}_{h,su}^{(1,1)}}{n^{3/2}},$$

$$L_{10} = \frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s+u}^{(2)}}{n^{3/2}}, L_{11} = \frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}_{h,s+u}^{(1)}\tilde{\mu}_{h,s}^{(2)}}{n^{3/2}}, L_{12} = \frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\tilde{\mu}_{h,suv}^{(1,1,1)}}{n^{3/2}}, L_{13} = \frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,(s+u)v}^{(1,1)}}{n^{3/2}},$$

$$L_{14} = \frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\tilde{\mu}_{h,(s+u)}^{(1)}\tilde{\mu}_{h,s(u+v)}^{(1,1)}}{n^{3/2}}, L_{15} = \frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\tilde{\mu}_{h,(s+u+v)}^{(1)}\tilde{\mu}_{h,su}^{(1,1)}}{n^{3/2}}, L_{16} = \frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,(s+u)}^{(1)}\tilde{\mu}_{h,(s+u+v)}^{(1)}}{n^{3/2}}.$$

Performing the necessary (tedious but straightforward) calculations and using the notation R_i, $\tilde{R}_i$, $i \in \{1,2,3....16\}$ with $\lim_{n\to\infty} R_i = \lim_{n\to\infty}\tilde{R}_i = 0$, we get:

$$L_1 = \begin{cases} c_{18}(c_4-1)^{-1}\frac{c_4^n}{n^{3/2}} + R_1 & \text{if } c_4 \neq 1 \\ \frac{\omega\left(\omega^2+3\omega\varphi\bar{h}+3\gamma c_1\right)}{2}\frac{n^2}{n^{3/2}} + \tilde{R}_1 & \text{if } c_4 = 1 \end{cases}, L_2 = R_2, L_3 = R_3,$$

$$L_4 = \begin{cases} \frac{\varphi c_{18}}{(1-c_4)(\varphi-c_4)}\frac{c_4^n}{n^{3/2}} + \frac{\bar{h}}{2}c_1\frac{n^2}{n^{3/2}} + R_4 & \text{if } c_4 \neq 1 \\ \frac{\bar{h}}{2}\left(c_1 + \varphi\left(\omega^2 + 3\bar{h}\varphi\omega + 3\gamma c_1\right)\right)\frac{n^2}{n^{3/2}} + \tilde{R}_4 & \text{if } c_4 = 1. \end{cases}$$

$$L_5 = \begin{cases} \frac{\gamma c_{18}}{(1-c_4)(\gamma-c_4)}\frac{c_4^n}{n^{3/2}} + c_1\frac{\bar{h}}{2}\frac{n^2}{n^{3/2}} + R_5 & \text{if } c_4 \neq 1 \\ \left[\begin{array}{l} \gamma(1-\gamma)^{-1}\frac{\omega\left(\omega^2+3\bar{h}\varphi\omega+3\gamma c_1\right)}{2} \\ \omega^2(1-\gamma)^{-1}\frac{\bar{h}}{2} + \omega\varphi\bar{h}^2 \end{array}\right]\frac{n^2}{n^{3/2}} + \tilde{R}_5 & \text{if } c_4 = 1 \end{cases}$$

$$L_6 = \frac{\bar{h}^3}{2}\frac{n^2}{n^{3/2}} + R_6, L_7 = \frac{\bar{h}^3}{2}\frac{n^2}{n^{3/2}} + R_7, L_8 = \frac{\bar{h}^3}{2}\frac{n^2}{n^{3/2}} + R_8, L_9 = \frac{\bar{h}^3}{2}\frac{n^2}{n^{3/2}} + R_9, L_{10} = \frac{c_1\bar{h}}{2}\frac{n^2}{n^{3/2}} + R_{10},$$

$$L_{11} = \frac{c_1\bar{h}}{2}\frac{n^2}{n^{3/2}} + R_{11}, L_{12} = \begin{cases} \begin{array}{l} \varphi\gamma c_{18}(c_4-1)^{-1}(c_4-\varphi)^{-1}(c_4-\gamma)^{-1}\frac{c_4^n}{n^{3/2}} + \frac{\bar{h}^3}{6}\frac{n^3}{n^{3/2}} \\ +\frac{\bar{h}}{2}(1-\varphi)^{-1}\left(2\varphi\left(c_1-\bar{h}^2\right) + \bar{h}\left(\left(h_{t+1}-\bar{h}\right)-\omega\right)\right)\frac{n^2}{n^{3/2}} + R_{12} \end{array} & \text{if } c_4 \neq 1 \\ \frac{\bar{h}^3}{6}\frac{n^3}{n^{3/2}} + \frac{\bar{h}}{2}\left[\begin{array}{l} (1-\varphi)^{-1} \\ \left(c_1(1+\varphi) - 2\varphi\bar{h}^2 + \bar{h}\left(\left(h_{t+1}-\bar{h}\right)-\omega\right)\right) \\ +\varphi\gamma(1-\gamma)^{-1}\left(\omega^2 + 3\omega\varphi\bar{h} + 3\gamma c_1\right) \end{array}\right]\frac{n^2}{n^{3/2}} + \tilde{R}_{12} & \text{if } c_4 = 1 \end{cases}$$

$$L_{13} = \frac{\bar{h}^3}{6}\frac{n^3}{n^{3/2}} + \frac{\bar{h}}{2}\left(-\bar{h}^2 + (1-\varphi)^{-1}\left(\bar{h}\left(h_{t+1}-\bar{h}\right) + \varphi\left(c_1-\bar{h}^2\right)\right)\right)\frac{n^2}{n^{3/2}} + R_{13},$$

$L_{14} = \frac{\bar{h}^3}{6}\frac{n^3}{n^{3/2}} + \frac{\bar{h}^2}{2}\left(-\bar{h} + \left(h_{t+1} - \bar{h}\right)(1-\varphi)^{-1}\right)\frac{n^2}{n^{3/2}} + R_{14},$

$L_{15} = \frac{\bar{h}^3}{6}\frac{n^3}{n^{3/2}} + \frac{\bar{h}}{2}(1-\varphi)^{-1}\left(-2\bar{h}^2 + \bar{h}h_{t+1} + c_1\varphi\right)\frac{n^2}{n^{3/2}} + R_{15},$

$L_{16} = \frac{\bar{h}^3}{6}\frac{n^3}{n^{3/2}} + \frac{\bar{h}^2}{2}\left(-\bar{h} + \left(h_{t+1} - \bar{h}\right)(1-\varphi)^{-1}\right)\frac{n^2}{n^{3/2}} + R_{16}.$

Performing the necessary calculations, we obtain the expression in (14), where:

$$N = \frac{\omega\left(\omega^2+3\omega\varphi\bar{h}+3\gamma c_1\right)}{2} + 3\frac{\bar{h}}{2}\left(c_1 + \varphi\left(\omega^2 + 3\omega\varphi\bar{h} + 3\gamma c_1\right)\right)$$
$$+3\left[\gamma(1-\gamma)^{-1}\frac{\omega\left(\omega^2+3\omega\varphi\bar{h}+3\gamma c_1\right)}{2} + \omega^2(1-\gamma)^{-1}\frac{\bar{h}}{2} + \omega\varphi\bar{h}^2\right]$$
$$+6\frac{\bar{h}}{2}\left[\begin{array}{l}(1-\varphi)^{-1}\left(c_1(1+\varphi) - 2\varphi\bar{h}^2 + \bar{h}\left(\left(h_{t+1}-\bar{h}\right)-\omega\right)\right)\\ +\varphi\gamma(1-\gamma)^{-1}\left(\omega^2+3\omega\varphi\bar{h}+3\gamma c_1\right)\end{array}\right] - 6\frac{\bar{h}}{2}(1-\varphi)^{-1}\varphi\left(2c_1 - \bar{h}^2\right)$$

For $\gamma \in (1,\infty)$, we can write: $\lim_{n\to\infty} \mathrm{T}_{h,n} = \frac{\lim_{n\to\infty}\left(\gamma^{-3/2n}M_{h,n}^{(3)}\right)}{\left[\lim_{n\to\infty}\left(\gamma^{-n}M_{h,n}^{(2)}\right)\right]^{3/2}},$

where $\lim_{n\to\infty}\left(\gamma^{-n}M_{h,n}^{(2)}\right) = -\left(\tilde{\mu}_{h,1}^{(2)} - c_3\right)(1-\gamma)^{-1} - 2\varphi(1-\varphi)^{-1}\left[\left(\tilde{\mu}_{h,1}^{(2)} - c_3\right)(1-\gamma)^{-1} - (\varphi-\gamma)^{-1}\right].$

For $\gamma = 1$, we can write: $\lim_{n\to\infty} \mathrm{T}_{h,n} = \frac{\lim_{n\to\infty}\left(n^{-3}M_{h,n}^{(3)}\right)}{\left[\lim_{n\to\infty}\left(n^{-2}M_{h,n}^{(2)}\right)\right]^{3/2}},$

where $\lim_{n\to\infty}\left(n^{-2}M_{h,n}^{(2)}\right) = \frac{1}{2}\left[2\varphi(1-\varphi)^{-1} + 1\right]\left(\omega^2 + 2\omega\varphi\bar{h}\right).$