Analytic Approximations to GARCH Aggregated Returns Distributions with Applications to VaR and ETL

Carol Alexander
ICMA Centre, University of Reading

Emese Lazar
ICMA Centre, University of Reading

Silvia Stanescu
Kent Business School, University of Kent

May 5, 2011

ICMA Centre Discussion Papers in Finance DP 2011-08

ICMA Centre • The University of Reading
Whiteknights • PO Box 242 • Reading RG6 6BA • UK
Tel: +44 (0)1183 788239 • Fax: +44 (0)1189 314741
Web: www.icmacentre.rdg.ac.uk
Director: Professor John Board, Chair in Finance
The ICMA Centre is supported by the International Capital Market Association

ABSTRACT

It is widely accepted that some of the most accurate predictions of aggregated asset returns are based on an appropriately specified GARCH process. As the forecast horizon is greater than the frequency of the GARCH model, such predictions either require time-consuming simulations or they can be approximated using a recent development in the GARCH literature, viz. analytic conditional moment formulae for GARCH aggregated returns. We demonstrate that this methodology yields robust and rapid calculations of the Value-at-Risk (VaR) generated by a GARCH process. Our extensive empirical study applies Edgeworth and Cornish-Fisher expansions and Johnson SU distributions, combined with normal and Student *t*, symmetric and asymmetric (GJR) GARCH processes to returns data on different financial assets; it validates the accuracy of the analytic approximations to GARCH aggregated returns and derives GARCH VaR estimates that are shown to be highly accurate over multiple horizons and significance levels.

JEL Code: C53, G17

Keywords: GARCH, higher conditional moments, approximate predictive distributions, Value-at-Risk, Conditional VaR, Expected tail loss, Expected shortfall

Carol Alexander
Chair of Risk Management
ICMA Centre, University of Reading,
Reading, RG6 6BA, UK.
Email: c.alexander@icmacentre.ac.uk

Emese Lazar
Lecturer in Finance,
ICMA Centre, University of Reading,
Reading, RG6 6BA, UK.
Email: e.lazar@icmacentre.ac.uk

Silvia Stanescu
Lecturer in Finance,
Kent Business School, University of Kent,
Canterbury, CT2 7PE, UK.
Email: s.stanescu@kent.ac.uk

1 INTRODUCTION

In an era when financial products can be extremely complex, sophisticated models for the density forecasting of portfolio returns are an important topic for academic research. Forward-looking returns distributions have a plethora of applications to portfolio risk assessment and allocation optimization, and accurate forecasts of the entire distribution are crucial if we believe that returns depart significantly from normality.

Given the widely documented characteristics of financial asset returns, quite complex dynamic models are needed for predicting distributions of underlying asset returns. A salient feature is their volatility clustering - that is, "large changes tend to be followed by large changes – of either sign – and small changes tend to be followed by small changes" (Mandlebrot, 1963). Generalised autoregressive conditional heteroscedastic (GARCH) models, introduced by Engle (1982), Bollerslev (1986) and Taylor (1986), have proved very successful in capturing this behaviour, and they can also explain why asset returns distributions are skewed and leptokurtic.

When aggregated returns are generated by a GARCH process, Engle (2003) argues in his Nobel lecture that simulations are required to predict the quantiles of the returns distribution over a time horizon which is longer than the frequency of the model. Simulations are only asymptotically exact and it can be very time consuming to simulate aggregated GARCH returns distributions to a satisfactory degree of accuracy. This computational burden will reduce the scope for out-of-sample tests of the predictive returns distributions. By the same token, any practical implementation of a GARCH model in portfolio risk assessment and/or optimization will be limited to over-night rather than intra-day calculations.

Hence, there is a clear need for fast and accurate analytic approximations to the returns distributions that would otherwise need to be simulated for various GARCH processes. This paper presents an empirical study of the effectiveness of the modelling framework suggested by Alexander, Lazar and Stanescu (2011) for generating GARCH aggregated returns distributions, with particular reference to the accuracy of lower quantiles that are used for estimates of portfolio Value-at-Risk (VaR).[1]

[1]Alexander, Lazar and Stanescu (2011), henceforth denoted by ALS, derived analytic formulae for the conditional moments of forward and aggregated GJR-GARCH returns, and forward and aggregated GJR-GARCH variances, up to order four, with a generic innovation process, thus encompassing a number of standard GARCH models. We shall utilize only a subset of their results, viz. their formulae for the moments of the aggregated returns distribution, under just four standard GARCH models, namely the symmetric GARCH(1,1) process and the asymmetric GJR-GARCH process, each

Since the 1996 Amendment to the Basel I Accord, VaR has become the standard metric for financial risk assessment and reporting, not only in the major banks that must now use VaR as a basis for their assessment of market risk capital reserves, but also in asset management, hedge funds, mutual funds, pension funds, corporate treasury and indeed in virtually every large institution worldwide that has dealings in the financial markets. As a result the academic literature on VaR is huge.[2] Some of the most influential academic research concerns the use of GARCH processes to measure VaR at the aggregate (“top-down") level, rather than utilizing standard (“bottom-up") VaR model for assessing a firm’s market risk capital. A path-breaking paper by Berkowitz and O’Brien (2002) utilizes aggregate profit and loss data from six of the world’s major banks to demonstrate a very clearly superior accuracy in top-down GARCH-based VaR estimates relative to more traditional, bottom-up VaR estimates.

Given the frequent turmoil in financial markets and the pervasive use of the VaR metric throughout the industry, the construction of fast, accurate and easily implemented VaR measures is not only timely – it is of great practical and regulatory importance. The contribution of our paper lies in that it applies moment-based approximation methods frequently used in the literature and/or practice, like the Cornish-Fisher expansion or Johnson SU distribution, in a new modellling framework, i.e. a GARCH VaR context, thus combining the accuracy of GARCH modelling with the speed of these approximation methods. Also, the paper provides rather extensive empirical tests of these approximations in a GARCH framework, using different statistical tests, data samples, horizons, significance levels and an out-of-sample period that includes the current crisis.[3]

First we apply the aggregated return moment formulae of ALS to three broad market risk factors: an equity index (S&P 500), a cross-currency pair (Euro/dollar), and a discount bond (3-month US Treasury bill). Then we apply standard distribution approximation methods to these moments (the Edgeworth expansion and a fit of the Johnson SU distribution) and evaluate their accuracy using the asymptotically exact simulated distributions as benchmark.[4] But the main focus of this paper is on

with normal and Student *t* error distributions.

[2]A condensed literature survey is provided in Section 3. More comprehensive reviews may be found in Alexander (2008), Angelidis and Degiannakis (2009) and Christoffersen (2009).

[3]In-sample size for GARCH model parameter estimation is 10 years of daily log returns; out-of-sample statistical tests cover a 10-year period from 3 January 2000 to 31 December 2009.

[4]We examine the proximity of each quasi-analytic distribution to the simulated distribution using both Kolmogorov-Smirnov (see Kolmogorov (1933), Smirnov (1939), Massey(1951)) and Cramer-von Mises tests (see Cramer (1928) and Anderson and Darling (1952)).

the speed and accuracy of our quasi-analytic VaR predictions, which are assessed using the coverage tests of Christoffersen (1998). The reminder of this paper is organised as follows: Section 2 presents the theoretical methodology that we shall implement for our empirical results; Section 3 reviews the VaR literature and explains how analytic formulae for the first four moments of aggregated GARCH returns can be used to approximate VaR; Section 4 presents the data and empirical results;[5] and Section 5 concludes.

2 Approximate Aggregated GARCH Returns Distributions

Our purpose is to approximate distributions of the aggregated returns in a GARCH framework that capture the important characteristics of financial asset returns, i.e. their volatility clustering and their non-normal distributions. Here we show how such approximate distributions can be obtained using analytic formulae for the first four conditional moments of GARCH aggregated returns.

Consider the following generic GJR specification, introduced by Glosten, Jagannathan and Runkle (1993), for the generating process of a continuously compounded portfolio return from time $t-1$ to time t, denoted r_t:

$$r_t = \mu + \varepsilon_t, \quad \varepsilon_t = z_t h_t^{1/2}, \quad z_t \sim D(0,1),$$

with
$$h_t = \omega + \alpha\,\varepsilon_{t-1}^2 + \lambda\,\varepsilon_{t-1}^2 I_{t-1}^- + \beta\,h_{t-1},$$

where $h_t = V\left(r_t\,|\Omega_{t-1}\right)$ is the variance of the portfolio return, conditional on the information set $\Omega_{t-1} = \{r_{t-j}, j \geq 1\}$. The GARCH error ε_t is a disturbance process and z_t is a sequence of *i.i.d.* zero mean unit variance random variables with distribution D. I_t^- is an indicator function which equals 1 if $\varepsilon_t < 0$ and zero otherwise. The symmetric GARCH(1,1) model can be obtained from the above by equating $\lambda = 0$. In our empirical results we shall allow $D\,(0,1)$ to be either a standard normal or a standardized Student *t* distribution, with degrees of freedom estimated by maximum likelihood along with the other GARCH model parameters. Thus we shall consider four different possibilities for the GARCH processes that are most appropriate for different types of asset returns, namely the normal and Student *t* GJR and GARCH(1,1) models.

Denote the first four central moments of the n-period future aggregated returns generated by the

[5] For convenience, the standard statistical that underpin these results are stated in an appendix.

model above as:

$$E_t\left[\left(R_{tn}-E_t\left(R_{tn}\right)\right)^i\right] \text{ for } i=1,2,3,4, \text{ where } R_{tn}=\sum_{s=1}^{n} r_{t+s}.$$

ALS (Theorem 1) derived exact formulae for both central and standardized moments of the aggregated returns. The conditional mean $\tilde{M}_{R,n}^{(1)}$, variance $M_{R,n}^{(2)}$, skewness $T_{R,n}$, and kurtosis $K_{R,n}$ of the n-period return are given by:

$$\tilde{M}_{R,n}^{(1)}=n\mu, \quad M_{R,n}^{(2)}=n\bar{h}+(1-\varphi)^{-1}\left(1-\varphi^n\right)\left(h_{t+1}-\bar{h}\right)$$

$$T_{R,n}=\left[\tau_z\sum_{s=1}^{n}E_t\left(h_{t+s}^{3/2}\right)+3\sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right)\right]\left(M_{R,n}^{(2)}\right)^{-3/2},$$

$$K_{R,n}=\left(\begin{array}{l}\kappa_z\sum\limits_{s=1}^{n}E_t\left(h_{t+s}^2\right)+\sum\limits_{s=1}^{n}\sum\limits_{u=1}^{n-s}\left(4E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^3\right)+6E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right)\right)\\ +12\sum\limits_{s=1}^{n}\sum\limits_{u=1}^{n-s}\sum\limits_{v=1}^{n-s-u}E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}^2\right)\end{array}\right)\left(M_{R,n}^{(2)}\right)^{-2},$$

with:

1. $\varphi=\alpha+\lambda F_0+\beta$, with F_0 being the distribution function for $D(0,1)$ evaluated at zero;
2. $\bar{h}=\omega(1-\varphi)^{-1}$, so if $\varphi\in(0,1)$, then $\bar{h}$ is the steady-state variance;
3. $E_t\left(h_{t+s}^{3/2}\right)\simeq\frac{5}{8}(E_t\left(h_{t+s}\right))^{3/2}+\frac{3}{8}E_t\left(h_{t+s}^2\right)\left(E_t\left(h_{t+s}\right)\right)^{-1/2}$, where
   $E_t\left(h_{t+s}\right)=\bar{h}+\varphi^{s-1}\left(h_{t+1}-\bar{h}\right)$, and $E_t\left(h_{t+s}^2\right)=c_1+\left(h_{t+1}^2-c_3\right)\gamma^{s-1}+c_2\varphi^{s-1}$
   with $\gamma=\varphi^2+\left(\kappa_z-1\right)\left(\alpha+\lambda F_0\right)^2+\kappa_z\lambda^2F_0\left(1-F_0\right)$,
   $c_1=\left(\omega^2+2\omega\varphi\bar{h}\right)(1-\gamma)^{-1}$, $c_2=2\omega\varphi\left(h_{t+1}-\bar{h}\right)(\varphi-\gamma)^{-1}$ and $c_3=c_1+c_2$.
4. $E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right)=\varphi^{u-1}\left(\alpha\tau_z+\lambda\int\limits_{z=-\infty}^{0}z^3f(z)\,dz\right)E_t\left(h_{t+s}^{3/2}\right)$, where f is the *pdf* of $D(0,1)$.
5. $E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^3\right)=\tau_zE_t\left(\varepsilon_{t+s}h_{t+s+u}^{3/2}\right)$, where

$$E_t\left(\varepsilon_{t+s}h_{t+s+u}^{3/2}\right)\simeq\tfrac{3}{4}c_4\left[\left(E_t\left(h_{t+s+u}\right)\right)^{1/2}+\omega\varphi(\varphi-\gamma)^{-1}\left(E_t\left(h_{t+s+u}\right)\right)^{-1/2}\right]\varphi^{u-1}E_t\left(h_{t+s}^{3/2}\right)$$
$$+\tfrac{3}{8}\left(E_t\left(h_{t+s+u}\right)\right)^{-1/2}\gamma^{u-1}\left(c_5E_t\left(h_{t+s}^{5/2}\right)+2\omega\gamma(\gamma-\varphi)^{-1}c_4E_t\left(h_{t+s}^{3/2}\right)\right),$$

$$c_4=\alpha\tau_z+\lambda\int\limits_{z=-\infty}^{0}z^3f(z)\,dz$$

$$c_5=\alpha\left(\alpha\mu_z^{(5)}+2\beta\tau_z\right)+\lambda\left(2\alpha+\lambda\right)\int\limits_{z=-\infty}^{0}z^5f(z)\,dz+2\lambda\beta\int\limits_{z=-\infty}^{0}z^3f(z)\,dz$$

$$E_t\left(h_{t+s}^{5/2}\right)\simeq\tfrac{1}{8}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{1/2}\left(15\tilde{\mu}_{h,s}^{(2)}-7\left(\tilde{\mu}_{h,s}^{(1)}\right)^2\right)$$

6. $E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right)=\bar{h}\left(1-\varphi^u\right)E_t\left(h_{t+s}\right)+\varphi^{u-1}\kappa_z\left(\alpha+\lambda F_0++\kappa_z^{-1}\beta\right)E_t\left(h_{t+s}^2\right)$
7. $E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}^2\right)=c_4\varphi^{v-1}E_t\left(\varepsilon_{t+s}h_{t+s+u}^{3/2}\right)$

Table 1 outlines the modifications to the above generic formulae needed for the normal and Student *t*

	Normal	Student *t*
F_0	$\frac{1}{2}$	$\frac{1}{2}$
τ_z	0	0
κ_z	3	$3\frac{\nu-2}{\nu-4}$
$\mu_z^{(5)}$	0	0
$\int\limits_{z=-\infty}^{0} z^3 f(z)\,dz$	$-\sqrt{\frac{2}{\pi}}$	$-\frac{2}{\sqrt{\pi}}\frac{(\nu-2)^{3/2}}{(\nu-1)(\nu-3)}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$
$\int\limits_{z=-\infty}^{0} z^5 f(z)\,dz$	$-4\sqrt{\frac{2}{\pi}}$	$-\frac{8}{\sqrt{\pi}}\frac{(\nu-2)^{5/2}}{(\nu-1)(\nu-3)(\nu-5)}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$

TABLE 1: ***Parameter values for the normal and Student t special cases***
Note: ν denotes the degrees of freedom of the Student t distribution; $\nu > k$ for the k-th moment of a Student t distribution to exist and be finite.

GJR special cases. The normal and Student *t* GARCH(1,1) can be obtained by equating $\lambda = 0$ in the formulae for the corresponding GJR models.

Following ALS we approximate the distribution of the n-period returns using its first four moments and three different approximation methods, i.e. the Cornish-Fisher expansion, the Edgeworth expansion and Johnson SU distributions.

Cornish and Fisher (1937) and Fisher and Cornish (1960) developed an asymptotic expansion for the quantile function of a probability distribution whose cumulants[6] (moments) are known in terms of the standard normal quantile function.[7] When only the first few cumulants are used, one obtains an approximation of the quantile function. The Cornish - Fisher approximation is popular in empirical applications mainly due to its speed and relative simplicity. While the approximation is expected to perform well in the vicinity of the normal, because it is a local approximation, increasing the order does not necessarily improve the error of the approximation. Moreover, the resulting quantile function is not necessarily monotonic as a function of the tail probability, and it suffers from tail behaviour problems - i.e. the approximation error increases at extreme quantiles.[8]

Somewhat similar to the Cornish-Fisher expansion, the Edgeworth expansion represents a method of approximating a density of interest around a base density, usually the standard normal density. It

[6]The cumulants represent an alternative to the moments of a probability distribution; while the cumulants set is equivalent to that of the moments, there are cases where stating the problem in terms of the cumulants rather than the moments may be preferred. The cumulants are defined by the cumulant generating function, which is equal to the natural logarithm of the moment generating function.

[7]Hill and Davis (1968) later generalized the expansion, by expressing the quantiles of the distribution in question in terms of the quantiles of a base distribution, which need not be the standard normal.

[8]See also Jaschke (2002).

belongs to the class of Gram-Charlier expansions (see Chebyshev (1860), Chebyshev (1890), Gram (1883), Charlier (1905) and Charlier (1906)), being a rearrangement of a Gram-Charlier A series. However, Gram-Charlier A series and Edgeworth series are only equivalent asymptotically; in empirical applications where finite order approximations are considered, they can differ significantly. The Edgeworth version has the theoretical advantage of being a true asymptotic expansion, i.e. the error of the approximation is controlled. However, it shares the monotonicity and convergence problems of the Cornish-Fisher expansion. The first few terms of the Edgeworth expansion are:

$$f_x(x) \simeq f_x^E(x) = \varphi(x) - \frac{\tau_x}{6}\varphi^{(3)}(x) + \frac{(\kappa_x - 3)}{24}\varphi^{(4)}(x) + \frac{\tau_x^2}{72}\varphi^{(6)}(x), \tag{1}$$

where $f_x^E(x)$ is the second-order Edgeworth approximation of the density of interest f_x, φ is the standard normal density and $\varphi^{(k)}$ is its k^{th} derivative, and τ_x and κ_x denote the skewness and kurtosis of f_x. For our purposes f_x will be the density of the normalised aggregated returns.

Finally, the third approximation method we use here, the Johnson SU distribution, differs from the previous two in that it is a proper distribution rather than an expansion. Johnson (1949) introduced three monotonic transformations from a variable x to a standard normal variable z, corresponding to three (Johnson) distributions.[9] The Johnson SU distribution considered in this paper is the most relevant for financial applications, since it is leptokurtic. A random variable x is said to follow a Johnson SU distribution if:[10]

$$x = \xi + \lambda \sinh\left(\frac{z-\gamma}{\delta}\right) \tag{2}$$

where z is a standard normal variable. Tuenter (2001) developed a very fast algorithm for the estimation of the four parameters δ, γ, λ and ξ. Specificly, using Tuenter's (2001) algorithm, we are matching the first four conditional moments of the n-period aggregated GARCH returns (detailed in Section (2) above) to the corresponding moments of a Johnson SU distribution. Although flexible, the main disadvantage of this approach is that a Johnson SU distribution is not guaranteed to exist for any set of mean, variance, skewness and (positive) excess kurtosis.

[9] For a characterization of the family of Johnson distributions see also Bowman and Shenton (1983).

[10] Here we follow the notation of Tuenter (2001) for the four parameters of the Johnson SU (JSU) distribution. However, parameters λ and γ of the JSU distribution should not be confused with the GJR-GARCH parameter λ or the constant γ used in Section 2.

3 Value-at-Risk

The $\alpha\%$ n-day VaR of a portfolio is minus the α-quantile of its n-day returns distribution, thus it is the loss that is anticipated with $(1-\alpha)\%$ confidence from holding an unmanaged portfolio over a risk horizon of n days.[11] Manganelli and Engle (2001) distinguish three major types of VaR models, according to the methods used to forecast a distribution for future returns of the portfolio:

1. Parametric VaR models, which are mainly represented by the Riskmetrics methodology (see J. P. Morgan, 1996), assume a particular approximation of the portfolio mapping function, e.g. a linear (delta) approximation, or a quadratic (delta-gamma) approximation. Analytic formulae for VaR estimates may be derived only when tractable, parametric distributions are used for risk factor returns. But these are often unrealistic and inaccurate. Instead Monte Carlo simulation must be applied. Efficient Monte Carlo methods were first proposed by Glasserman, Heidelberger and Shahabuddin (2001) and many other researchers since, because this type of simulation can be extremely time-consuming, as noted below.
2. Non-parametric VaR models are essentially represented by historical simulation. Perignon and Smith (2010) note that this is the most widely-used approach, based on a survey of major banks around the world. In a more sophisticated system this is often augmented with a GARCH model, such as in the filtered historical simulation methodology introduced by Barone-Adesi *et al.* (1998, 1999).[12] Alexander and Sheedy (2008) demonstrate that historical simulation is highly inaccurate without such additional filtering.
3. Semi-parametric VaR models include applications of extreme value theory (see Danielsson and deVries (1998) for example); and applications of linear and non-linear regression quantile techniques, as in Taylor (1999), Chernozhukov and Umantsev (2001), and Engle and Manganelli (2004). Techniques based on quasi-maximum likelihood GARCH, developed in Bollerslev and Woolridge (1992), also fall into this category. Other semi-parametric VaR models combine the above approaches – see Manganelli and Engle (2001) and McNeil and Frey (2000).

[11] We employ the standard notation α for the the quantile of the aggregated returns distribution; this should not be confused with the α parameter of the GARCH models.

[12] See also Boudoukh, Richardson and Whitelaw (1998), for an alternative filtering approach.

The approach we propose falls into the category of parametric VaR models, and is closely related to a sub-stream of the academic research on VaR. As we detail below, we estimate VaR from alternative approximations of the distribution of returns based on moments. Zangari (1996) was the first to introduce a parametric method for estimating VaR based on higher moments, which he called Modified VaR; this can be thought of as an estimator for VaR that corrects the baseline Gaussian VaR for skewness and kurtosis. The "correction" is done using a Cornish-Fisher expansion. Following Zangari (1996), the Cornish-Fisher expansion was also applied for quantile estimation by Mina and Ulmer (1999), Favre and Galeano (2002), Amenc, Martellini and Vaissie (2003), Gueyie and Amvella (2006), Qian (2006), Boudt, Peterson and Croux (2009) and Simonato (2010). Favre and Galeano (2002), Amenc, Martellini and Vaissie (2003) and Gueyie and Amvella (2006) all use the Modified VaR in a portfolio optimisation setting, while Qian (2006) employs it in a risk budgeting application. Mina and Ulmer (1999) compare four alternative methods for constructing an approximate delta-gamma portfolio distribution, namely Johnson distributions, Cornish-Fisher expansion, Fourier transforms (for the moment generating function) and partial Monte Carlo. Also related to this research, Wong and So (2003) approximate the distribution of QGARCH[13] aggregated returns with a skewed Student *t* distribution based on moments and subsequently derive a corresponding VaR measure. Their model encompasses a general GARCH(1,1) process with various innovation distributions, but it does not encompass the GJR model that is used in this paper. Also, their chosen approximation method, the skewed Student *t* distribution, is different from those we employ here. Boudt, Peterson and Croux (2009) derive a Modified conditional VaR – also called expected tail loss (ETL) – as an application of the Edgeworth expansion, while Simonato (2010) considers VaR and ETL measures derived for Cornish-Fisher and Cram-Charlier expansions and Johnson distribution approximations, in the context of Merton's (1976) model.

An $\alpha\%$ n-day VaR estimate is derived from the α-quantile of the n-period portfolio return distribution as:

$$\mathrm{VaR}_{n,\alpha,t} = -\hat{F}^{-1}_{t;t+n}(\alpha)\,, \text{ or equivalently as } \int_{-\infty}^{-VaR_{n,\alpha,t}} \hat{f}_{t;t+n}(x)\,dx = \alpha \qquad (3)$$

[13]See Engle (1990), Sentana (1991) and Campbell and Hentschel (1992).

where $\hat{F}^{-1}_{t;t+n}$ is the time t forecast of the distribution function for returns aggregated from time t to time $t+n$, and $\hat{f}_{t;t+n}$ is the corresponding density function. The corresponding ETL is given by:[14]

$$\mathrm{ETL}_{n,\alpha,t} = -\frac{1}{\alpha} \int_{x=-\infty}^{-VaR_{n,\alpha,t}} x \hat{f}_{t;t+n}(x)\, dx \tag{4}$$

The purpose of this section is to present analytic approximations for (3) and (4) based on the first four moments of aggregated GARCH returns. Some of the VaR formulae (though not the ETL) are quite well-established; indeed they have been applied to VaR modelling by several authors and we briefly reviewed the contributions of some of these authors above. However, they have never before been applied in the GARCH framework. Thus, given these moment-based VaR and ETL formulae, we use the results for the conditional moments of aggregated returns in a GARCH context to derive analytic approximations for GARCH VaR and ETL purely in terms of the estimated GARCH model parameters.

Using (2), one can immediately write the expression for the Johnson SU VaR as:

$$\mathrm{VaR}^{\mathrm{JSU}}_{n,\alpha,t} = -\lambda \sinh\left(\frac{z_\alpha - \gamma}{\delta}\right) - \xi, \tag{5}$$

where $z_\alpha = \Phi^{-1}(\alpha)$ is the lower α-quantile of the standard normal distribution. The formula for the Johnson SU ETL, derived in Simonato (2010) (and adapted to our setting) is:

$$\mathrm{ETL}^{\mathrm{JSU}}_{n,\alpha,t} = -\alpha^{-1}\left(\xi\Phi(y) + \frac{\lambda}{2}e^{\left(-\frac{\gamma}{\delta}+\frac{1}{2\delta^2}\right)}\Phi\left(y-\frac{1}{\delta}\right) - \frac{\lambda}{2}e^{\left(\frac{\gamma}{\delta}+\frac{1}{2\delta^2}\right)}\Phi\left(y+\frac{1}{\delta}\right)\right) \tag{6}$$

where $y = \gamma + \delta \sinh^{-1}\left(\frac{\mathrm{VaR}^{\mathrm{JSU}}_{n,\alpha,t}-\xi}{\lambda}\right)$.

Truncating the terms beyond the fourth cumulant, the expression for the Cornish-Fisher VaR as a function of the first four standardized moments of the n-day aggregated returns is:

$$\mathrm{VaR}^{\mathrm{CF}}_{n,\alpha,t} = -\left[z_\alpha + \frac{\mathrm{T}_{R,n}}{6}\left(z_\alpha^2 - 1\right) + \frac{(\mathrm{K}_{R,n}-3)}{24} z_\alpha\left(z_\alpha^2-3\right) - \frac{\mathrm{T}^2_{R,n}}{36} z_\alpha\left(2z_\alpha^2 - 5\right)\right]\sqrt{M^{(2)}_{R,n}} - \tilde{M}^{(1)}_{R,n} \tag{7}$$

To express the Cornish-Fisher ETL one would need to perform an inversion of the Cornish-Fisher expansion, which is given in terms of the quantile (or inverse distribution) function. To our knowledge,

[14] The definitions in (3) and (4) are stated for continuous distribution (density) functions. Acerbi and Tasche (2002) give definitions for VaR and ETL that also apply to discontinuous distribution functions.

no such inversion was derived. One other possibility would be to integrate the expression in (7) after $\mathrm{VaR}^{\mathrm{CF}}_{n,\alpha,t}$; however such an approach is deemed to be inaccurate, as the Cornish-Fisher VaR is less accurate when we go further into the tail and is likely to be very inaccurate when $\alpha \to 0$.

4 Empirical Methodology and Results

4.1 Data

The performance of our proposed quasi-analytical distribution forecasts and VaR methodologies is tested using equity index (S&P 500), foreign exchange (Euro/dollar) and interest rate (3-month Treasury bill) daily data. These three series represent three major market risk types (equity, foreign exchange and interest rate risk, respectively) and within each class they represent the most important risk factors in terms of volumes of exposures. The three data sets used in this application were obtained from Datastream and each comprise 20 years of daily data, from 1st January 1990 to 31st December 2009.[15] Figure 1 plots the daily log returns for the equity and exchange rate data and the daily changes in the interest rate.[16] Table 2 presents the sample statistics of the empirical unconditional distribution returns. In accordance with stylized facts on daily financial returns the mean of every series is not statistically different from zero and the unconditional volatility is highest for equity and lowest for interest rates.[17] Skewness is negative and low (in absolute value) but significant for all three series, so that extreme negative returns are more likely than extreme positive returns of the same magnitude, while excess kurtosis is always positive and highly significant, suggesting that the unconditional distributions of the series have more probability mass in the tails than the normal distribution. We notice that the interest rate sample exhibits the most significant departures from normality, while the Euro/dollar series is the closest to normality among the three we analyze.

[15] Since the Euro was only introduced in 1999, the ECU/dollar exchange rate is used for the period between 1990 and 1999.

[16] First differences in fixed maturity interest rates are the equivalent of log returns on corresponding bonds.

[17] The standard error of the sample mean is equal to the (sample) standard deviation, divided by sample size and we assumed 252 trading days per year to annualize the standard deviation into volatility.

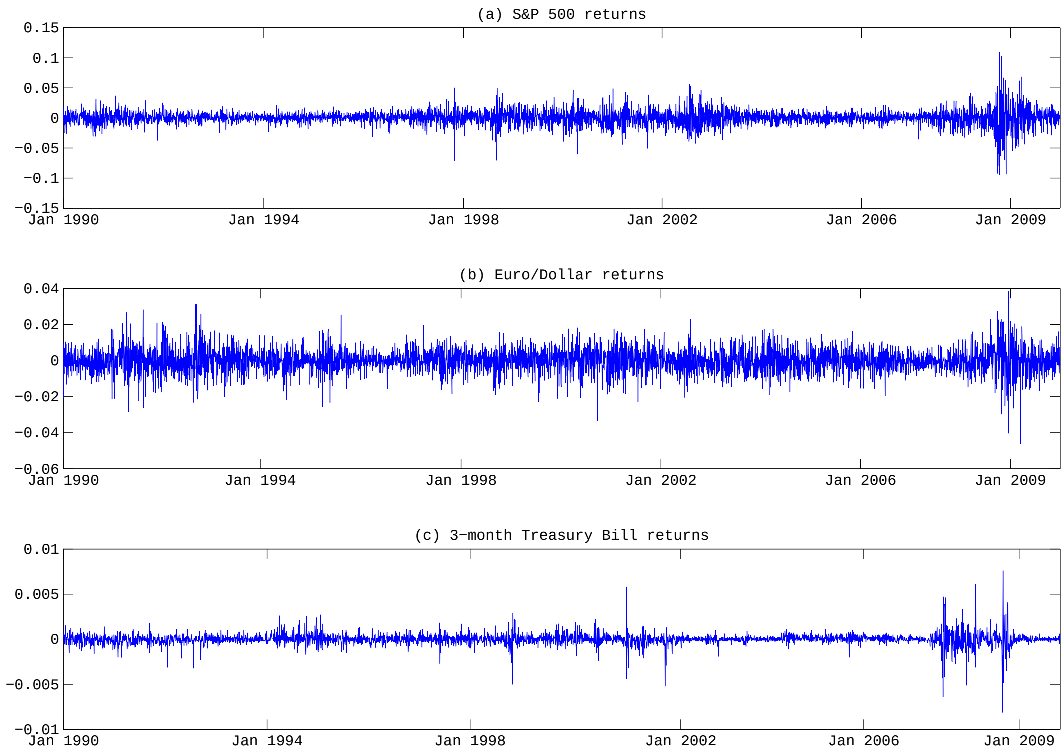

FIGURE 1: ***Returns*** *The equity and exchange rate daily returns are computed as the first differences in the log of the S&P500 index values and Euro/dollar exchange rates, respectively. The interest rate returns are computed as first differences in interest rate values. All returns are computed over the period 1st January 1990 to 31st December 2009.*

4.2 Empirical Methodology

Four different GARCH models, namely the baseline GARCH(1,1) and the asymmetric GJR, each with normal and Student t error distributions, are estimated for each of the three time series.[18] The estimation is conducted in a rolling window format, where a window of ten years of daily data (window size approximately 2500 observations) is rolled daily for an additional ten years. The resulting time series of model parameters are subsequently used to estimate the first four conditional moments of aggregated returns based on the analytic formulae from Section 2, from 3rd January 2000 to 31st December 2009, for three time horizons: $n = 5, 10, 20$ working days, respectively. For the symmetric models – the normal and Student t GARCH(1,1) – the skewness is zero by construction. However,

[18]Based on the BIC and AIK information criteria, an AR(4) model was used to remove the autocorrelation in the data for the 3-month Treasury bill sample, while for the S&P 500 sample an AR(2) suffices to remove all autocorrelation in the returns; in what follows, estimation and testing is based on the residuals from these regressions for the two samples. No autocorrelation was found in the foreign exchange data.

	S&P 500	EUR/USD	3M IR
Mean	0.00022	-2E-05	-2E-05
Maximum	0.1	0.0384	0.0076
Minimum	-0.0947	-0.0462	-0.0081
Volatility	0.1862	0.1	0.0101
Skewness	-0.1981***	-0.0992**	-0.5007***
Excess kurtosis	9.1643***	2.7136***	28.2342***

TABLE 2: ***Summary statistics.*** *The summary statistics are of the equity and exchange rate daily log returns, and of the daily changes in interest rates from 1990 to 2009. Asterisks denote significance at 0.05 (*), 0.01 (**) and 0.001(***). The standard error of the sample mean is equal to the (sample) standard deviation, divided by sample size. The standard errors under the null of normality are approximately* $(6/T)^{1/2}$ *and* $(24/T)^{1/2}$*for the skewness and kurtosis, respectively, where T is the sample size. We used 252 risk days per year to annualize the standard deviation into volatility.*

the asymmetric specifications – the normal and Student t GJR – lead to non-zero skewness estimates. All four models yield positive excess kurtosis for all horizons and all time series.

4.3 Distribution Tests

For the implementation of the distribution tests described in the Appendix we combine the four GARCH specifications (the normal and Student t GARCH(1,1) and GJR models) with two approximation methods (the Johnson SU distribution and the Edgeworth expansion) and thus have eight alternative approximate (theoretical) distributions to evaluate and compare with the corresponding simulated distributions (based on 10000 simulations).[19] From approximately 2500 sets of parameter estimates and corresponding moments estimates, the tests are performed for 150 days from a low volatility period (January to August 2006), 150 days from a high volatility period (August 2008 to March 2009) and the last 150 observations from 2009. In Table 3 these periods are labelled 'low vol', 'high vol' and 'current' respectively. Finally, the time horizon we consider here is n = 5 days.

Table 3 summarizes the results of the Kolmogorov-Smirnov (KS) and Cramer - von Mises (CVM) tests for each of the eight approximate distributions considered. We report the mean values and the associated standard deviations of the KS distance D and CVM test statistic and also the percentage of times when the computed test statistics were higher than the corresponding asymptotic 5% critical value. Since we perform the tests at the 5% significance level we expect a 5% rejection

[19]For the interest rate sample, fitting the Johnson SU distribution was problematic and hence, for this sample, we do not report results for the Johnson SU Student t GJR for either the distribution tests or the VaR coverage tests.

rate. The 5% critical values are 0.0136 for the KS distance and 0.416 for the CVM test statistic.[20] Although the asymptotic critical values do not apply exactly in our case, the model that produces the lowest values of the test statistic is still the best among the alternatives.[21]

The results in Table 3 generally show that the proposed moment-based distributions successfully approximate the distributions of aggregated returns obtained via GARCH simulations. When departures from normality are less significant – as in the case of the Euro/dollar exchange rates – both approximation methods and all GARCH models yield similar and very good results. However, the more significant the departures from normality, the greater the differences between the results produced by the two approximation methods. Thus, for the S&P 500 sample, the results produced by the two approximation methods are still very similar for the normal models and comparable for the Student t models. However, for the interest rate sample, the Johnson SU distribution, when it exists, produces superior results to its Edgeworth counterpart.[22] In the great majority of cases the KS and CVM tests agree on model rankings. While the KS test always yields better results (lower average test statistics) when the Johnson SU distribution rather than the Edgeworth expansion is employed, for the same GARCH model, in a few cases the CVM sometimes slightly favours the Edgeworth methodology (e.g. Normal GARCH(1,1) in the 'current' sub-period, for the S&P 500 sample).

[20] These are asymptotic results for a test where the distribution being tested for is continuous, fully known and generic (no particular family of distributions assumed). Stephens (1970) derives modified statistics for the finite sample case; however, with a sample size of 10000, these modifications are not actually needed and the asymptotic results would apply, if the hypothetical distribution were fully specified. However, in our case this distribution is based on estimated results and hence the above mentioned critical values do not apply and we would need to simulate the correct critical values if we were to properly carry out the tests. Still, we report the percentage of times the test statistics are greater than the asymptotic critical values, so that we can infer, approximately, if the test results are at least in the vicinity of these asymptotic critical values. We also note that the results have to be interpreted with care since it is likely that the appropriate (simulated) critical values for this testing exercise are lower than the asymptotic critical values reported above.

[21] What we mean by "best among alternatives" here is "closest to the (respective) simulated distribution". However, one has to interpret the results with care since the simulated distribution is obviously not the same for all alternative approximate distributions.

[22] For the S&P 500 sample, the percentage differences between the values of test statistics obtained with the Johnson SU and the Edgeworth expansion (for the same GARCH model) are no higher than 10%, and even lower for the Euro/dollar sample. However, the values of the distribution test statistics differ by a factor of at least 2.5 between the Johnson SU and the Edgeworth expansion for the interest rate sample.

4.4 VaR Estimation and Backtesting

We evaluate the accuracy of the proposed VaR estimates over 5, 10 and 20-day risk horizons using the coverage tests of Christoffersen (1998) described in the Appendix.[23] We combine the four different GARCH models with two approximation methods, the Johnson SU distribution and the Cornish-Fisher expansion, and derive the VaR estimates for each GARCH model, and for each approximation method, and for α = 10%, 5%, 1% and 0.1%.

Tables 4 - 6 summarize the results of the likelihood ratio (LR) tests for the unconditional coverage, independence and conditional coverage of predictive intervals for log returns (or, in the case of Treasury Bill rates, absolute changes) aggregated over horizons of n = 5, 10 and 20 working days. In these tables (*), (**), or (***) denote a result that is statistically significant at 10%, 5% and 1%, respectively, i.e. the null of correct coverage is rejected. Also, if no value is reported for the independence test, that is because there are no consecutive violations, and obviously the models in question pass the test.

(a) S&P 500

The results in Table 3 show that the model that performs best across all horizons, significance levels and approximation methods is the normal GJR, which incurs no rejections in the coverage tests for this sample. For the 10-day horizon, the Student t GJR also performs extremely well, incurring no rejections in the coverage tests. Also, none of the models is rejected in the independence test for this sample, across all horizons and significance levels. For the 5-day horizon, we notice that while there are inter-model differences in terms of the test results obtained for different GARCH specifications, the results obtained by combining the same GARCH model with different approximation methods are either very similar (for the normal models) or slightly better better with the Johnson SU approximation. For example, for the 10% VaR estimates when a Student t GARCH model is employed, the results obtained with the Johnson SU are better (lower LR_{uc} and LR_{cc} test statistics) than with the corresponding Cornish-Fisher estimates. Finally, we can argue that the coverage tests results are indeed good for the proposed methodologies, bearing in mind that these are out-of-sample, forecasting results.

[23]To avoid using over-lapping observations, as this would violate the independence assumption for the indicator process in the unconditional coverage test, we use only every n-th set of parameter/moments estimates, where n is either 5, 10 or 20 working days.

(b) Euro/dollar

The results further improve for the Euro/dollar sample, the sample with the least significant non-normality features. Now both the normal GARCH(1,1) and the normal GJR yield no rejections in the coverage tests, across all horizons, significance levels and approximation methods. Furthermore, their Student t counterparts also yield very few rejections. Again, none of the models is rejected in the independence test, across all horizons and significance levels.

(c) 3-month Treasury Bill

The 3-month Treasury Bill sample is the one exhibiting the most pronounced non-normalities among the three samples we analyze. As for the distribution tests reported above, for this sample our methodology performs slightly less well than it does for the other two samples, especially for the longer horizons. For the 5-day horizon we find that the normal GARCH(1,1) produces no rejections in the coverage tests across all significance levels and approximation methods. The performance of the normal GJR is also good, being only marginally rejected in the independece test for the 5% VaR. For the 10- and 20- day horizon, no model performs perfectly; however, the performance of the normal GJR remains relatively good, especially when coupled with the Johnson SU distribution. Out of the three samples we analyze, this is the only sample for which the models are sometimes rejected in the independence tests. Also for this sample, the superior performance of the Johnson SU over the expansion method is more apparent, especially if we compare the results obtained for the 10% VaR.

To give an example of the speed of our methodology relative to Monte Carlo simulation, on a PC with Intel i5-650 (dual core) and 4Gb RAM using Excel 2010 VBA, the time recorded for computing Student t GJR-GARCH VaR estimates for a 10-day horizon using our quasi-analytic methodology was only 0.254 seconds. By comparison, to compute the 10-day VaR based on 10,000 Monte Carlo simulation took 13 seconds. It would be greater for VaR computions over longer horizons. Moreover, 10,000 is typically regarded as the minimum number of simulations to be used for a passable degree of accuracy, and the time would be extrapolated linearly as the number of simulations increases.

5 CONCLUSIONS

This paper demonstrates empirically that approximate aggregated GARCH return distributions can be accurately constructed based on a number of alternative approximation methods. Subsequently, we focused on quasi-analytic GARCH VaR measures constructed from analytic formulae for higher moments and the accuracy of our results shows that time-consuming simulations are no longer needed for GARCH VaR estimation.

Based on their occurrence in the related literature and on the feasibility of obtaining fast, analytical formulae for the distribution function and/or the associated VaRs and ETLs, we selected three alternative approximation methods based on analytic moments. A comprehensive testing exercise used three different samples on three major sources of market risk - equity (S&P 500), foreign exchange (Euro/dollar) and interest rate risk (3-month Treasury bill). We first tested how close the approximate distributions constructed using the Johnson SU distribution and the Edgeworth expansion are to their simulated counterparts. Consistently good results were obtained for the S&P 500 and the Euro/dollar exchange rate, but the Johnson SU is superior for the T-Bill where non-normalities are highly significant.

We then tested the accuracy of our methodology for VaR estimation using the likelihood ratio tests for conditional coverage, proposed by Christoffersen (1998). Here we combine the Cornish-Fisher expansion and the Johnson SU distribution with four GARCH specifications (normal and Student t GARCH(1,1) and GJR) which results in eight alternative VaR models to test and compare. VaR is estimated at four significance levels (0.1%, 1%, 5%, and 10%) and for three different time horizons (5, 10 and 20 days). Our quasi-analytic GARCH VaR estimates are extremely accurate, especially for the S&P 500 and Euro/dollar samples where departures from normality are less significant. When models are rejected in the statistical testing, it is generally due to inappropriate unconditional coverage and very rarely due to rejections in the independence tests. In fact, especially at the higher confidence levels, the models very often yield no consecutive violations. The results are even more remarkable if we consider that the analysis is entirely out-of-sample and that the testing period (2000-2009) contains the Dotcom 'bubble burst' (2000) and out of the ten years of out-of-sample data three years cover the current financial crisis (2007-2009).

References

Acerbi, C. and Tasche, D. (2002) Expected Shortfall: A Natural Coherent Alternative to Value at Risk. *Economic Notes* 31(2), 379-388.

Alexander, C. (2008) *Market Risk Analysis*, Volume IV: *Value at Risk Models*. Wiley, Chichester.

Alexander, C., Lazar, E., Stanescu, S. (2011) Analytic Moments of GARCH Processes. *Working Paper*, available from http://ssrn.com.

Alexander, C. and Sheedy, E. (2008) Developing a Stress Testing Framework based on Market Risk Models. *Journal of Banking and Finance* 31:10, 2220 - 2236.

Amenc, N., Martellini, L., and Vaissie, M. (2003) Benefits and Risks of Alternative Investment Strategies. *Journal of Asset Management* 4(2), 96-118.

Anderson, T.W. and Darling, D.A. (1952) Asymptotic Theory of Certain "Goodness of Fit" Criteria Based on Stochastic Processes. *The Annals of Mathematical Statistics* 23(2), 193-212.

Angelidis, T. and Degiannakis, S. (2009) Econometric Modelling of Value-at-Risk. In W.N. Toggins (ed.), *New Econometric Modelling Research*. Nova, New York.

Artzner,P., Delbaen F., Eber, J.M. and Heath, D. (1999) Coherent Measures of Risk. *Mathematical Finance* 9, 203- 228.

Barone-Adesi, G., Bourgoin, F. and Giannopoulos, K. (1998) Don't Look Back. *Risk* 11, 100-103.

Barone-Adesi, G., Giannopoulos, K. and Vosper, L. (1999) VaR without Correlations for Nonlinear Portfolios. *Journal of Futures Markets* 19, 583-602.

Berkowitz, J., and O'Brien, J. (2002). How Accurate are Value-at-Risk Models at Commercial Banks? *Journal of Finance*. 55, 1093-1111.

Bollerslev, T. (1986) Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics* 31, 307-327.

Bollerslev, T. and Woolridge J.M. (1992) Quasi-Maximum Likelihood Estimation and Inference in Dynamic Models with Time Varying Covariance. *Econometric Reviews* 11, 143-172.

Boudt, K., Peterson, B. and Croux, C. (2009) Estimation and Decomposition of Downside Risk for Portfolios with Non-Normal Returns. *Journal of Risk* 11(2), 189-200.

Bowman, K.O. and Shenton, L.R. (1983) Johnson's System of Distributions. In S. Kotz and N. L. Johnson (Eds), *Encyclopedia of Statistical Sciences*, vol. 4, pp. 303-314. New York, Wiley.

Boudoukh, J., Richardson, M. and Whitelaw, R. (1998) The Best of Both Worlds, *RISK* 11, 64-67.

Campbell, J.Y. and Hentschel, L. (1992) No News is Good News: an Asymmetric Model of Changing Volatility in Stock Returns. *Journal of Financial Econometrics* 31, 281-318.

Charlier, C. V. L. (1905) Uber das Fehlergesetz. *Arkiv for Matematik, Astronomi Och Fysic* 2 (8).

Charlier, C. V. L. (1906) Uber die Darstellung willkurlicher Funktionen. *Arkiv for Matematik, Astronomi Och Fysic* 2(20).

Chebyshev, P. L. (1860) Sur le Developpment des Fonctions a une Seule Variable. *Bulletin de la classe physique-mathematique de l'Academie Imperiale des Sciences St. Petersbourg* 3 (1), 193-202.

Chebyshev, P. L. (1890) Sur Deux Theorems Relatifs aux Probabilites. *Acta Mathematica* 14, 305-315.

Chernozhukov, V. and Umantsev, L. (2001) Conditional Value-at-Risk: Aspects of Modelling and Estimation. *Empirical Economics* 26, 271-293.

Christoffersen, P.F. (1998) Evaluating Interval Forecasts. *International Economic Review* 39 (4), 841-862.

Christoffersen, P. F. (2009) Value-at-Risk Models. In T. Mikosch, J.P. Kreiss, R.A. Davis and T.G. Ander (Eds). *Handbook of Financial Time Series*. Springer, Berlin.

Christoffersen, P.F. and Diebold, F. X. (2000) How Relevant is Volatility Forecasting for Financial Risk Management? *Review of Economics and Statistics* 82, 1-11.

Clements, M.P. and Taylor, N. (2003) Evaluating Interval Forecasts of High-Frequency Financial Data. *Journal of Applied Econometrics* 18, 445-456.

Cornish, E.A. and R.A. Fisher (1937) Moments and Cumulants in the Specification of Distributions. *Review of the International Statistical Institute* 5, 307-320.

Cramer, H. (1928) On the Composition of Elementary Errors. *Skandinavisk Aktuarielidskrift* 11, 141-180.

Danielsson, J. and de Vries, C. (1998) Beyond the Sample: Extreme Quantile and Probability Estimation. Mimeo, *Tinbergen Institute Rotterdam*.

Engle, R.F. (1982) Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica* 50(4), 987-1007.

Engle, R.F. (1990) Discussion: Stock Market Volatility and the Crash of '87. *Review of Financial Studies* 3, 103-106.

Engle, R.F. (2003) Risk and Volatility: Econometric Models and Financial Practice. *Nobel Lecture*, available at http://nobelprize.org.

Engle, R.F. and Manganelli, S. (2004) CAViaR: Conditional Autoregressive Value at Risk by Regression Quantile. *Journal of Business and Economic Statistics* 22(4), 367-381.

Favre, L., and Galeano, J.A. (2002) Mean-Modifed Value-at-Risk Optimization with Hedge Funds. *Journal of Alternative Investments* 5(2), 2-21.

Fisher, R.A. and Cornish, E.A. (1960) The Percentile Points of Distributions Having Known Cumulants. *Technometrics* 2, 209-226.

Glasserman, P., Heidelberger P. and Shahabuddin, P. (2001) Efficient Monte Carlo Methods for Value-at-Risk. In *Mastering Risk* Volume 2: *Applications*. Prentice Hall, Great Britain.

Glosten, L.R., Jagannathan, R. and Runkle, D.E. (1993) On the Relation Between the Expected Value and the Volatility of the Nominal Excess Return on Stocks. *Journal of Finance* 48(5), 1779-1801.

Gram, J. P. (1883) Uber die Entwickelung Reeler Funktionen in Reihen mittelst der Methode der Kleinsten Quadrate. *Creile* 94, pp. 41-73.

Gueyie, J.P., and Amvella, S.P. (2006) Optimal Portfolio Allocation Using Funds of Hedge Funds. *Journal of Wealth Management* 9(2), 85-95.

Hill, G.W. and Davis, A. W. (1968) Generalized Asymptotic Expansions of Cornish-Fisher Type. *The Annals of Mathematical Statistics* 39(4), 1264 - 1273.

J.P. Morgan (1996) *RiskMetrics Technical Document*, 4th Edition. J.P. Morgan, New York.

Jaschke, S. (2002) The Cornish-Fisher-Expansion in the Context of Delta-Gamma-Normal Approximations. *Journal of Risk* 4(4), 33-52.

Johnson, N.L. (1949) Systems of Frequency Curves Generated by Methods of Translation. *Biometrica* 36,149-76.

Kolmogorov, A. (1933) Sulla Determinazione Empirica di una Legge di Distribuzione. *Giornale dell'Instituto Italiano degh Attuari* 4, 1-11.

Kupiec, H. (1995) Techniques for Verifying the Accuracy of Risk Management MModels. *Journal of Derivatives* 3, 73-84.

Malevergne, Y. and Sornette, D. (2003) Testing the Gaussian Copula Hypothesis for Financial Assets Dependences. *Quantitative Finance*, 3, 231-250.

Mandelbrot, B. (1963) The Variations of Certain Speculative Prices. *Journal of Business* 36, 394-419.

Manganelli, S. and Engle, R. F. (2001) Value-at-Risk Models in Finance. *European Central Bank*

Working Paper Series No.75.

Massey, F.J. Jr (1951) The Kolmogorov-Smirnov Test for Goodness of Fit. *Journal of the American Statistical Association* 46 (253), 68-78.

McNeil, A.J. and Frey, R. (2000) Estimation of Tail-Related Risk Measures for Heteroscedastic Financial Time Series: an Extreme Value Approach. *Journal of Empirical Finance* 7, 271-300.

Merton, R.C. (1976) Option Pricing when the Underlying Process for the Stock Returns is Discountinuous. *Journal of Financial Economics* 3, 124-144.

Mina, J. and Ulmer, A. (1999) Delta-Gamma Four Ways. *RiskMetrics Group*.

Perignon, C., and Smith, D. R. (2010) The Level and Quality of Value-at-Risk Disclosure by Commercial Banks. *Journal of Banking and Finance* 34, 362-377.

Qian, E. (2006) On the Financial Interpretation of Risk Contribution: Risk Budgets Do Add up. *Journal of Investment Management* 4(4), 1-11.

Scheffe, H. (1943) Statistical Inference in the Non-Parametric Case. *Annals of Mathematical Statistics* 14, 305-332.

Sentana, E. (1991) Quadratic ARCH Models: a Potential Reinterpretation of ARCH Models as Second-Order Taylor Approximations. *Working Paper*, London School of Economics and Political Science, London.

Simonato, J.G. (2010) The Performance of Johnson Distributions for Value at Risk and Expected Shortfall Computation, *Working Paper*.

Smirnov, N. (1939) Sur les Ecarts de la Courbe de Distribution Empirique. *Matematicheskii Sbornik* 48(6), 3-26.

Taylor, J. (1999) A Quantile Regression Approach to Estimating the Distribution of Multi-Period Returns. *Journal of Derivatives*, 64-78.

Taylor, S. (1986) *Modeling Financial Time Series*. Wiley, New York.

Tuenter, H. J. H. (2001) An Algorithm to Determine the Parameters of SU-Curves in the Johnson System of Probability Distributions by Moment Matching. *Journal of Statistical Computation and Simulation* 70(4), 325-347.

Wolfowitz, J., (1949) Non-Parametric Statistical Inference. *Proceedings of the Berkeley Symposium of Mathematical Statistics and Probability*, University of California Press: Berkeley.

Wong, C. M. and So, M. K. P. (2003) On Conditional Moments of GARCH Models with Applications to Multiple Period Value at Risk Estimation. *Statistica Sinica* 13, 1015-1044.

Zangari, P. (1996) A VaR Methodology for Portfolios that Include Options. *RiskMetrics Monitor*, JP Morgan-Reuters, First Quarter, 4-12.

APPENDIX: EVALUATION METHODS

We first investigate whether the proposed approximate distributions provide an adequate representation of the conditional distributions of aggregated returns. Since these distributions are not observable, not even ex-post, we use simulated distributions as proxies. Specifically, we test whether $F_m = F_s$, where F_m is the cumulative distribution function (*cdf*) for the approximate distribution of aggregated returns constructed using the first four conditional moments and the Edgeworth expansion or Johnson SU distribution approximation methods, and F_s is the simulated *cdf* for the n-day GARCH aggregated returns. F_s is given by the step-function of the sample: $F_s(x_i) = T^{-1}i$, where x_i with $i \in \{1, 2, ..., T\}$ is the increasingly ordered simulated sample, i = number of returns less than or equal with x_i, and T is the sample size (number of simulations).

To test whether $F_m = F_s$, we employ two well known distribution tests: the Kolmogorov-Smirnov (KS), proposed by Kolmogorov (1933), Smirnov (1939), Scheffe (1943) and Wolfowitz (1949) and the Cramer - von Mises (CVM), proposed by Cramer (1928).[24]. The KS test statistic is given by: KS $= \sqrt{T}D$, where D is the maximum distance between the two distributions, i.e. $D = \max_x |F_m(x) - F_s(x)|$, whereas the CVM (or $n\omega^2$) criterion is defined as: CVM $= T\int_x (F_m(x) - F_s(x))^2 dF_m(x)$. For practical implementations and for an (increasingly) ordered sample, simpler variants of the the KS and CVM test statistics are given by:

$$\text{KS} = \sqrt{T}\max_{1\le i\le T}\left\{\max\left[F_m(x_i) - \frac{i-1}{T}, \frac{i}{T} - F_m(x_i)\right]\right\},$$

$$\text{CVM} = \sum_{i=1}^{T}\left[F_m(x_i) - \frac{2i-1}{2T}\right]^2 + \frac{1}{12T},$$

Finally, these statistics only have standard distributions if the distribution under the null hypothesis is entirely pre-specified, but in our case the F_m distribution relies on estimated parameter values so the theoretical critical values are no longer applicable.

To evaluate the accuracy of our quasi-analytic GARCH VaR we apply the statistical tests based on VaR exceedances that have become standard in the applied financial economics literature: the coverage tests for VaR accuracy of Christoffersen (1998) and others. The LR test, introduced by Kupiec (1995) and extended by Christoffersen (1998), is the most frequently used statistical tool for evaluating the performance of VaR models. Kupiec (1995) proposed two likelihood ratio (LR) tests:

[24] For generalizations of both the KS and CVM tests see Anderson and Darling (1952)

one based on times until the first failure (TUFF) and another LR test based on the proportion of failures (PF). However, as Kupiec himself acknowledges, both tests (and especially the TUFF version) have relatively low power in small samples. Hence, unless a large sample is available for model assessment purposes, these LR tests will have little power in deeming a VaR methodology inappropriate. Kupiec's (1995) PF test was later extended by Christoffersen (1998).[25] Christoffersen's conditional coverage LR test ($LR_{cc;\alpha}$) asserts that a good VaR model is one that produces a series of indicator functions

$$\{I_{t,\alpha}\}_{t=1}^{T} \equiv \left\{I_{R_{tn}<-\mathrm{VaR}_{n,\alpha,t}}\right\}_{t=1}^{T}$$

which are Bernoulli(α) i.i.d. regardless of whether R_{tn} is serially correlated and/or heteroskedastic. Christoffersen (1998) proved that:

$$LR_{cc;\alpha} = LR_{uc;\alpha} + LR_{ind;\alpha},$$

where $LR_{uc;\alpha}$ tests for the correct unconditional coverage, given that $\{I_t\}_{t=1}^{T}$ is independent, while $LR_{ind;\alpha}$ tests for the independence of this series.[26] He also derives the following test statistics and their respective distributions under the null to make the concepts operational:

$$LR_{uc;\alpha} = -2\ln\left(\left(\frac{1-\alpha}{1-\pi_\alpha}\right)^{n_{0;\alpha}}\left(\frac{\alpha}{\pi_\alpha}\right)^{n_{1;\alpha}}\right) \sim \chi^2(1)$$

$$LR_{ind;\alpha} = -2\ln\left(\frac{(1-\pi_{2;\alpha})^{n_{00;\alpha}+n_{10;\alpha}}\pi_{2;\alpha}^{n_{01;\alpha}+n_{11;\alpha}}}{(1-\pi_{01;\alpha})^{n_{00;\alpha}}\pi_{01;\alpha}^{n_{01;\alpha}}(1-\pi_{11;\alpha})^{n_{10;\alpha}}\pi_{11;\alpha}^{n_{11;\alpha}}}\right) \sim \chi^2(1)$$

$$LR_{cc;\alpha} = LR_{uc;\alpha} + LR_{ind;\alpha} \sim \chi^2(2)$$

where the ML estimates of the test statistics - $\hat{L}R_{uc;\alpha}$, $\hat{L}R_{ind;\alpha}$, and $\hat{L}R_{cc;\alpha}$ - are obtained for:

$$\hat{n}_{1;\alpha} = \sum_{t=1}^{T}\hat{I}_{t,\alpha};\quad \hat{n}_{0;\alpha} = T - \hat{n}_{1;\alpha};\quad \hat{\pi}_\alpha = \frac{\hat{n}_{1;\alpha}}{T};\quad \hat{n}_{ij;\alpha} = \sum_{t=1}^{T}\hat{J}_{ij;t;\alpha}$$

$$\hat{\pi}_{01;\alpha} = \frac{\hat{n}_{01;\alpha}}{\hat{n}_{00;\alpha}+\hat{n}_{01;\alpha}};\quad \hat{\pi}_{11;\alpha} = \frac{\hat{n}_{11;\alpha}}{\hat{n}_{10;\alpha}+\hat{n}_{11;\alpha}};\quad \hat{\pi}_{2;\alpha} = \frac{\hat{n}_{01;\alpha}+\hat{n}_{11;\alpha}}{\hat{n}_{00;\alpha}+\hat{n}_{10;\alpha}+\hat{n}_{01;\alpha}+\hat{n}_{11;\alpha}}$$

[25] Christoffersen (1998) actually proposes a very general, model-free approach to evaluating interval forecasts; Value-at-Risk forecasting is in effect a special case of interval forecasting, where the forecasting interval is one-sided. Moreover, although developed in a VaR context, Kupiec's test is not restricted to evaluating VaR models either: it can as well be applied to evaluating interval forecasts: the test for unconditional coverage proposed by Christoffersen (1998) is in fact Kupiec's PF test.

[26] One drawback of Christoffersen's (1998) methodology is that independence of failures is only tested against a first order Markov dependence; hence the LR test for independence proposed by Christoffersen is unable to reject a VaR methodology as being inappropriate even if the failure series exhibits some sort of dependence, but this is not of first order Markov type. Christoffersen and Diebold (2000) and Clements and Taylor (2003) generalize this. However, their approach is also not flawless, since the practical implementation estimation of their regression based testing procedure is often troublesome, especially for extreme quantile estimates.

$$\hat{I}_{t,\alpha} = I(r_{tn} < -\mathrm{VaR}_{n,\alpha,t}); \quad \hat{J}_{ij;t;\alpha} = I(\hat{I}_{t-1,\alpha} = i \cap \hat{I}_{t,\alpha} = j); \quad i,j = 0,1$$

where T is the sample size and r_{tn} are the sample realizations of the random variable R_{tn}.

$LR_{uc;\alpha}$ is essentially a simple hypothesis test: the null hypothesis is that the difference between the empirical exceedance rate and the desired level α is zero.[27] This null is then tested against the alternative that the exceedance rate is significantly higher or lower than the desired α. The test will thus discard methods as being inappropriate either because they tend to produce too little or too few exceedances, regardless of their timing. By also taking into account the clustering of exceedances, as well as the number of times the VaR is exceeded, $LR_{cc;\alpha}$ is a joint test of correct coverage and independence of hits, i.e. correct conditional coverage.

[27] An exceedance occurs when the return is lower than minus VaR.

		Normal GARCH(1,1)				Student *t* GARCH(1,1)				Normal GJR				Student *t* GJR			
		total	low vol	high vol	current	total	low vol	high vol	current	total	low vol	high vol	current	total	low vol	high vol	current
S&P 500		Johnson SU															
	KS-average	0.0113	0.0085	0.0086	0.0169	0.0088	0.0086	0.009	0.0088	0.009	0.0088	0.0091	0.0092	0.0097	0.0094	0.0098	0.0099
	KS-stdev	0.0144	0.0025	0.0024	0.0238	0.0026	0.0025	0.0026	0.0027	0.0026	0.0024	0.0027	0.0027	0.0028	0.0027	0.0028	0.0029
	KS-rejections@5%	8.67%	4.67%	4.67%	16.67%	5.78%	6.00%	5.33%	6.00%	5.78%	3.33%	6.00%	8.00%	9.78%	8.00%	9.33%	12.00%
	CVM-average	1.1472	0.1554	0.1608	3.1253	0.1686	0.1592	0.174	0.1726	0.1796	0.1704	0.1818	0.1867	0.2127	0.1957	0.2206	0.2218
	CVM-stdev	5.8194	0.1351	0.1396	9.8033	0.1408	0.1397	0.1382	0.145	0.1498	0.1398	0.1587	0.1509	0.1716	0.1546	0.1811	0.1778
	CVM-rejections@5%	9.33%	4.00%	5.33%	18.67%	4.67%	4.67%	4.00%	5.33%	5.78%	7.33%	4.67%	5.33%	7.56%	6.67%	6.67%	9.33%
		Edgeworth															
	KS-average	0.0114	0.0086	0.0087	0.017	0.0107	0.0102	0.0111	0.0108	0.0095	0.0095	0.0095	0.0096	0.0113	0.0112	0.0113	0.0113
	KS-stdev	0.0142	0.0025	0.0025	0.0235	0.0028	0.0028	0.0028	0.0029	0.0027	0.0026	0.0028	0.0028	0.003	0.0028	0.0031	0.0031
	KS-rejections@5%	9.56%	6.00%	4.67%	18.00%	15.33%	11.33%	17.33%	17.33%	8.44%	8.00%	6.67%	10.67%	18.67%	14.00%	19.33%	22.67%
	CVM-average	1.1322	0.1595	0.1643	3.0728	0.27	0.2354	0.2904	0.2844	0.2071	0.2041	0.2075	0.2096	0.3208	0.3049	0.3289	0.3287
	CVM-stdev	5.7089	0.1372	0.1389	9.6171	0.175	0.1659	0.1692	0.1853	0.1703	0.1602	0.1822	0.1689	0.2293	0.2061	0.243	0.2378
	CVM-rejections@5%	8.89%	4.67%	4.00%	18.00%	13.33%	9.33%	16.00%	14.67%	7.11%	6.67%	6.00%	8.67%	18.89%	13.33%	20.67%	22.67%
Euro/dollar		Johnson SU															
	KS-average	0.0086	0.0084	0.0086	0.0087	0.0087	0.0085	0.0087	0.0087	0.0086	0.0084	0.0086	0.0087	0.0087	0.0085	0.0088	0.0087
	KS-stdev	0.0025	0.0025	0.0024	0.0027	0.0026	0.0025	0.0025	0.0028	0.0025	0.0025	0.0024	0.0027	0.0026	0.0025	0.0025	0.0028
	KS-rejections@5%	4.67%	4.67%	4.67%	4.67%	4.44%	3.33%	4.00%	6.00%	4.44%	4.00%	4.67%	4.67%	4.89%	4.00%	4.00%	6.67%
	CVM-average	0.1625	0.156	0.1617	0.1697	0.166	0.1591	0.1645	0.1743	0.1625	0.1561	0.1613	0.1703	0.1663	0.159	0.1654	0.1745
	CVM-stdev	0.1401	0.1381	0.1411	0.1416	0.1423	0.1391	0.1404	0.1478	0.1401	0.1382	0.1402	0.1424	0.142	0.1398	0.1392	0.1475
	CVM-rejections@5%	5.11%	4.67%	4.67%	6.00%	4.67%	4.00%	4.67%	5.33%	5.11%	4.67%	4.67%	6.00%	4.22%	3.33%	4.00%	5.33%
		Edgeworth															
	KS-average	0.0086	0.0084	0.0086	0.0087	0.009	0.0089	0.009	0.009	0.0086	0.0084	0.0086	0.0087	0.009	0.0089	0.0091	0.0091
	KS-stdev	0.0025	0.0025	0.0024	0.0027	0.0027	0.0025	0.0025	0.0029	0.0025	0.0025	0.0024	0.0027	0.0026	0.0026	0.0025	0.0028
	KS-rejections@5%	5.11%	4.67%	4.67%	6.00%	6.00%	6.67%	4.00%	7.33%	4.67%	4.67%	4.67%	4.67%	6.44%	6.67%	4.67%	8.00%
	CVM-average	0.1626	0.1559	0.1617	0.1703	0.1784	0.1722	0.173	0.1899	0.1627	0.156	0.1612	0.171	0.1793	0.172	0.1756	0.1904
	CVM-stdev	0.1402	0.1382	0.1408	0.1422	0.1474	0.1448	0.1407	0.1567	0.1403	0.1384	0.1395	0.1434	0.1468	0.1458	0.1396	0.1549
	CVM-rejections@5%	5.11%	4.67%	4.67%	6.00%	4.89%	4.00%	4.67%	6.00%	4.89%	4.67%	4.67%	5.33%	4.44%	4.00%	4.00%	5.33%
3M Bill		Johnson SU															
	KS-average	0.0087	0.0084	0.0087	0.0089	0.0155	0.0144	0.0158	0.0165	0.0106	0.0094	0.0112	0.0113	N/A			
	KS-stdev	0.0025	0.0024	0.0024	0.0025	0.0036	0.0042	0.0031	0.0031	0.003	0.0026	0.003	0.0032				
	KS-rejections@5%	4.73%	4.00%	4.67%	5.69%	72.81%	59.33%	78.00%	82.93%	15.37%	5.33%	18.00%	24.39%				
	CVM-average	0.1614	0.152	0.1645	0.169	0.6838	0.6	0.686	0.7834	0.2575	0.1948	0.2889	0.2956				
	CVM-stdev	0.1288	0.1233	0.1349	0.1282	0.3202	0.3659	0.265	0.2943	0.1931	0.1431	0.2116	0.2044				
	CVM-rejections@5%	4.26%	4.00%	4.67%	4.07%	76.83%	60.67%	81.33%	91.06%	15.60%	6.00%	18.67%	23.58%				
		Edgeworth															
	KS-average	0.0235	0.0179	0.0269	0.026	0.0636	0.0511	0.0747	0.0655	0.0289	0.0214	0.0335	0.0326	0.0543	0.0468	0.0627	0.0533
	KS-stdev	0.0054	0.003	0.0041	0.0033	0.0123	0.0069	0.0094	0.0032	0.0071	0.0033	0.0054	0.004	0.0096	0.0065	0.0089	0.0037
	KS-rejections@5%	92.44%	95.33%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
	CVM-average	1.8418	0.9387	2.3913	2.273	17.198	10.55	23.437	17.696	2.9942	1.4219	3.9511	3.7446	11.682	8.5839	15.458	10.857
	CVM-stdev	0.8681	0.3213	0.7176	0.5385	6.6817	2.872	5.6867	1.5538	1.4736	0.4556	1.2011	0.8734	4.2451	2.5548	4.32	1.248
	CVM-rejections@5%	97.64%	93.33%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%

TABLE 3: ***Distribution tests for the approximate distributions of 5-day aggregated returns***

We report the average KS distance and CVM test statistic, with asociatted standard deviations and the percentage of cases where the test statistics are greater than the asymptotic 5% CVs(reject) for the 5-day aggregated returns for the S&P 500, Euro/dollar and 3-month Treasury Bills, respectively.

		Cornish-Fisher VaR				Johnson SU VaR			
Signif. level	Coverage test	Normal GARCH(1,1)	Student t GARCH(1,1)	Normal GJR	Student t GJR	Normal GARCH(1,1)	Student t GARCH(1,1)	Normal GJR	Student t GJR
					$n = 5$				
0.10%	LRuc	5.7431**	0.3828	0.3828	0.3828	5.7431**	2.5377	0.3828	0.3828
	LRind	-	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-	-
1%	LRuc	0.1819	0	0.225	0.225	0.1819	0.1818882	0.225	0.225
	LRind	-	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-	-
5%	LRuc	3.0046*	3.0046*	2.3918	4.4179**	3.0046*	3.0046*	2.3918	4.4179**
	LRind	1.2069	0.2214	-	1.427	1.2069	0.2214	-	1.427
	LRcc	4.2115	3.226	-	5.8449*	4.2115	3.226	-	5.8449*
10%	LRuc	8.5990***	12.008***	2.433	4.4786**	8.5990***	10.239***	2.433	3.9130**
	LRind	0.0009	0.008	0.0167	0.0755	0.0009	0.0334	0.0167	0.1437
	LRcc	8.5999**	12.009***	2.4497	4.5541	8.5999**	10.273***	2.4497	4.0567
					$n = 10$				
0.10%	LRuc	4.8160**	1.2684	-	-	4.8160**	4.8160**	1.2688	-
	LRind	-	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-	-
1%	LRuc	1.9365	0.757	0.0909	0.0909	1.9366	0.757	0.757	0.0909
	LRind	-	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-	-
5%	LRuc	0.1703	0.4751	0.5845	0.5845	0.1703	0.4751	0.5845	0.5845
	LRind	0.062	0.0122	-	-	0.062	0.0122	-	-
	LRcc	0.2323	0.4873	-	-	0.2323	0.4873	-	-
10%	LRuc	2.5398	5.5539**	0.0355	0.6445	2.5398	3.9140**	0.0355	0.6445
	LRind	0.0394	0.552	0.2463	0.8003	0.0394	0.3182	0.2463	0.8003
	LRcc	2.5792	6.1059**	0.2817	1.4448	2.5792	4.2322	0.2817	1.4448
					$n = 20$				
0.10%	LRuc	2.415	2.415	-	-	2.415	2.415	-	-
	LRind	-	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-	-
1%	LRuc	0.3846	0.0542	0.0542	0.0542	0.3846	0.0542	0.0542	0.0542
	LRind	-	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-	-
5%	LRuc	9.4252***	9.4252***	0.4757	4.4388**	9.4252***	9.4252***	1.1278	5.9327**
	LRind	0.3801	0.3801	0.6181	0.0088	0.4123	0.4123	0.2956	0.0694
	LRcc	9.8054***	9.8054***	1.0938	4.4476	9.8376***	9.8376***	1.4233	6.0022**
10%	LRuc	9.5294***	11.101***	1.6369	3.294*	9.5294***	11.101***	1.6369	3.294*
	LRind	0.0585	0.1932	0.0232	0.2732	0.0712	0.234	0.0232	0.309
	LRcc	9.5879***	11.294***	1.6601	3.5672	9.6006***	11.3348***	1.6602	3.603

TABLE 4: ***Coverage tests for the S&P 500***
Christoffersen's (1998) likelihood ratio tests for correct conditional coverage for the S&P 500 returns at horizons $h = 5$, 10 and 20 working days. Rejections of the null - of correct coverage - are marked with (*), (**) and (***) for the 10%, 5% and 1% significance levels, respectively.

		Cornish-Fisher VaR				Johnson SU VaR			
Signif. level	Coverage test	Normal GARCH(1,1)	Student t GARCH(1,1)	Normal GJR	Student t GJR	Normal GARCH(1,1)	Student t GARCH(1,1)	Normal GJR	Student t GJR
					$n = 5$				
0.10%	LRuc	0.3614	0.3614	0.3614	0.3614	0.3614	0.3614	0.3614	0.3614
	LRind	-	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-	-
1%	LRuc	0.272	5.0233**	0.272	5.0233**	0.272	1.05	0.272	1.05
	LRind	-	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-	-
5%	LRuc	0.0736	0.2204	0.005	0.4427	0.0736	0.2204	0.005	0.4427
	LRind	-	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-	-
10%	LRuc	0.1555	2.3401	0.062	1.5634	0.1555	1.231	0.062	0.6818
	LRind	1.7371	0.3537	1.5089	1.7885	1.7371	1.5516	1.5089	1.1266
	LRcc	1.8926	2.6946	1.5709	3.3519	1.8926	2.7826	1.5709	1.8084
					$n = 10$				
0.10%	LRuc	1.2393	1.2393	1.2393	1.2393	1.2393	1.2393	1.2393	1.2393
	LRind	-	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-	-
1%	LRuc	0.6985	0.1338	0.6985	1.2496	0.6985	0.0724	0.6985	0.1338
	LRind	-	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-	-
5%	LRuc	0.3781	1.3211	0.3781	1.9853	0.3781	1.3211	0.3781	1.9853
	LRind	-	0.0187	-	0.0713	-	0.0187	-	0.0713
	LRcc	-	1.3397	-	2.0566	-	1.3397	-	2.0566
10%	LRuc	1.6606	5.0297**	2.1987	5.0297**	1.6606	4.2234**	2.1987	4.2234**
	LRind	0.2987	0.099	0.1581	0.099	0.2987	0.216	0.1581	0.216
	LRcc	1.9593	5.1287*	2.3568	5.1287*	1.9593	4.4394	2.3568	4.4394
					$n = 20$				
0.10%	LRuc	-	-	-	-	-	-	-	-
	LRind	-	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-	-
1%	LRuc	1.694	1.694	1.694	0.3492	1.694	1.694	1.694	1.694
	LRind	-	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-	-
5%	LRuc	0.9927	1.8334	0.9927	0.9927	0.9927	1.8334	0.9927	0.9927
	LRind	-	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-	-
10%	LRuc	0.8303	4.9913**	1.4031	3.9112**	0.8303	4.9913**	1.4031	3.9112**
	LRind	0.7843	0.0952	1.1387	2.5828	0.7843	0.0952	1.1387	2.5828
	LRcc	1.6146	5.0865*	2.5419	6.4940**	1.6146	5.0865*	2.5419	6.4940**

TABLE 5: ***Coverage tests for the Euro/dollar***
Christoffersen's (1998) likelihood ratio tests for correct conditional coverage for the Euro/dollar returns at horizons $h = 5, 10$ *and* 20 *working days. Rejections of the null - of correct coverage - are marked with (*), (**) and (***) for the 10%, 5% and 1% significance levels, respectively.*

		Cornish-Fisher VaR				Johnson SU VaR		
Signif. level	Coverage test	Normal GARCH(1,1)	Student *t* GARCH(1,1)	Normal GJR	Student *t* GJR	Normal GARCH(1,1)	Student *t* GARCH(1,1)	Normal GJR
				$n = 5$				
0.10%	LRuc	0.6245	-	0.6245	-	0.6245	0.6245	0.6245
	LRind	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-
1%	LRuc	0.2191	0.0002	0.2917	1.2678	1.8122	0.2191	0.0002
	LRind	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-
5%	LRuc	1.1453	2.8861*	0.0704	2.2255	1.1453	1.1453	0.0012
	LRind	1.2412	0.5734	3.5535*	0.7648	1.2412	1.2412	3.0615*
	LRcc	2.3865	3.4594	3.6238	2.9903	2.3865	2.3865	3.0626
10%	LRuc	1.9663	15.582***	0.0025	12.346***	0.5892	0.0787	0.3079
	LRind	1.7918	4.5501**	0.303	6.4018**	0.8893	3.1424*	0.813
	LRcc	3.7581	20.132***	0.3055	18.748***	1.4785	3.2211	1.1208
				$n = 10$				
0.10%	LRuc	5.6086**	1.6141	1.6141	1.6141	10.660***	10.660***	5.6086**
	LRind	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-
1%	LRuc	0.4278	1.5452	0.4278	0.4278	3.1782*	1.5452	1.5452
	LRind	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-
5%	LRuc	1.4639	1.4639	0.0003	0.376	1.4639	1.4639	0.0003
	LRind	3.3584*	3.3584*	3.0584*	1.838	3.3584*	3.3584*	3.0584*
	LRcc	4.8222*	4.8222*	3.0587	2.214	4.8222*	4.8222*	3.0587
10%	LRuc	3.9066**	9.0401***	0.1942	7.864***	1.2417	1.7781	0.1942
	LRind	0.9642	3.9818**	0.1659	4.7915**	1.3039	0.9307	0.1659
	LRcc	4.8708*	13.022***	0.3601	12.656***	2.5457	2.7088	0.3601
				$n = 20$				
0.10%	LRuc	2.8133*	-	2.8133	-	2.8133*	2.8133*	2.8133*
	LRind	-	-	-	-	-	-	-
	LRcc	-	-	-	-	-	-	-
1%	LRuc	2.6323	0.7827	0.7827	0.7827	5.1822**	8.2582***	2.6324
	LRind	-	-	-	-	2.3722	-	-
	LRcc	-	-	-	-	7.5544**	-	-
5%	LRuc	1.6158	7.5402***	0.753	2.7510*	1.6158	5.7332**	1.6158
	LRind	0.2014	1.7605	0.48	1.5878	0.2014	0.5492	0.2014
	LRcc	1.8173	9.3007***	1.233	4.3388	1.8173	6.2825**	1.8173
10%	LRuc	23.070***	10.564***	30.7327***	7.3220***	3.4493*	2.447	0.9226
	LRind	5.9748**	2.1677	14.9262***	2.1151	0.9994	1.6011	1.1274
	LRcc	29.045***	12.732***	45.6590***	9.4372***	4.4487	4.0481	2.05

TABLE 6: ***Coverage tests for the 3mo IR***

Christoffersen's (1998) likelihood ratio tests for correct conditional coverage for the 3-month Treasury Bill returns at horizons $h = 5$, 10 and 20 working days. Rejections of the null - of correct coverage - are marked with (*), (**) and (***) for the 10%, 5% and 1% significance levels, respectively.