A General Approach to Real Option Valuation with Applications to Real Estate Investments

Carol Alexander and Xi Chen

First Version: January 23, 2012

ICMA Centre Discussion Papers in Finance: DP2012-04

ICMA Centre • The University of Reading
Whiteknights • PO Box 242 • Reading RG6 6BA • UK
Tel: +44 (0)1183 788239 • Fax: +44 (0)1189 314741
Web: www.icmacentre.rdg.ac.uk
Director: Professor John Board, Chair in Finance
The ICMA Centre is supported by the International Capital Market Association

ABSTRACT

We model investment opportunities with a single source of uncertainty, i.e. the market price of the investment. Investment cost can be predetermined or perfectly correlated with the market price. The common paradigm for risk-neutral real-option pricing is a special case encompassed within our general framework, and we analyse the relationship between standard real option prices and the more general risk-averse real option values. Numerical examples illustrate how these general values depend on the frequency of decision opportunities, the investor's risk tolerance and its sensitivity to wealth, his expected return and volatility of the underlying asset, and the price of the asset relative to initial wealth. Specific applications to real estate include property investment under 'boom-bust' or mean-reverting price scenarios, and buy-to-let or land-development opportunities.

Keywords: CARA, CRRA, certain equivalent, development, divestment, displaced log utility, exponential utility, HARA, investment, mean-reversion, property boom, real option, risk aversion, risk tolerance

JEL Classification: C44, D81, G13, G30

Carol Alexander,
Chair of Financial Risk Management,
ICMA Centre, Henley Business School at Reading
Email: c.alexander@icmacentre.ac.uk

Xi Chen,
PhD Student,
ICMA Centre, Henley Business School at Reading
Email:x.chen@icmacentre.ac.uk

Acknowledgements: Many thanks for many insightful discussions and very helpful comments from Prof. Charles Ward and Dr. Jacques Pezier, both of the ICMA Centre and to seminar participants at St. Gallen University.

1 Introduction

The term 'real option' is commonly applied to a decision opportunity for which the investment cost is predetermined, and the vast majority of the literature assumes the underlying asset is traded in a complete or partially complete market so that all (or at least the important) risks are hedgeable. Real options are typically regarded as tradable contracts with predetermined strikes, the standard risk-neutral valuation (RNV) principal is invoked, most commonly in a continuous-time setting, and the mathematical problem is no different to pricing an American option under the risk-neutral measure.

However, a financial option is only a special case of real option. The original definition of a real option, first stated by Myers (1977), is a decision opportunity for a corporation or an individual. It is a right, rather than an obligation, whose value is contingent on the uncertain price(s) of some underlying asset(s). The assumptions of hedgeable risks and predetermined strike are clearly inappropriate for many real options, particularly those in real estate, research and development or mergers and acquisitions. These markets are fundamentally different from the market for a liquid financial asset. Anything traded in a liquid market with no restrictions, e.g. a share, can always be transacted at the market price so an option to do so has no intrinsic value. By contrast, the market for the underlying of a real option is very often incomplete, the underlying investment can be highly illiquid and the transacted price is typically negotiated between individual buyers and sellers, e.g. via an auction. In this case Grasselli (2011) has proved that the real option to invest does have an intrinsic value, but is purely subjective and, unlike the premium on a financial option, it has no absolute accounting value. It represents the dollar amount, net of financing costs, that the investor should receive for certain to obtain the same utility value as the expected utility of the risky investment.

Real option values allow alternative investment opportunities to be ranked, just as financial investments are ranked using risk-adjusted performance measures. Thus, a pharmaceutical company may compare the values of real options to develop alternative products, or an oil exploration company may compare the values of drilling in different locations, or a private property development company may compare the values of opportunities to buy and develop plots of land in different locations.

This paper introduces a general decision-tree approach for determining the value of an investment opportunity and its optimal exercise path, in market that need not be complete because the solution is derived via maximization of expected utility. We endow the decision maker with a utility from the hyperbolic absolute risk aversion (HARA) class, introduced by Mossin (1968) and Merton (1971). Risk-neutrality is considered as a special case corresponding to infinite risk tolerance, i.e. a linear utility. Given the flexible utility structure we do not focus on analytic solutions, available only in the exponential case, and neither do we consider expansion approximations, preferring instead the direct numerical implementation of a very flexible framework.

We develop an approach that is both general in scope and flexible in its application to a wide range of investment or divestment decisions. Several numerical examples focus on real estate investments, from large-scale land development to individual residential property transactions. When we assume none of the risk of the project can be hedged in a financial market our framework is applicable to decisions faced by private companies, charitable entities or individuals. Such decisions can have profound implications for the decision maker's welfare. Housing may represent a major component of individual wealth and should not be viewed simply in expected net present value terms, nor should all uncertainties be based on systematic risk because they are largely unhedgeable. Private development companies may be generating returns and risks that have a utility value that is specific to the owners' outlook. Similarly, charitable entities such as housing trusts may have objectives that are far removed from wealth maximisation under risk-neutrality.

We provide a coherent analytical framework for computing the relative values of different real options under discrete decisions, for example at a monthly or quarterly meeting of the board of directors. The decision maker's utility function is applied to value every opportunity but the decision maker's subjective views about the evolution of the market price and any future rents or development costs are specific to each location. A variety of processes for a forward market price under the investor's (subjective) physical measure could be employed. Here we consider only one-factor models: geometric Brownian motion (GBM), mean-reversion and boom-bust market price scenarios. Extensions to jumps in prices and stochastic volatility are an interesting topic for further research.

An option strike can be predetermined, as is commonly assumed in the real option literature, or the investment cost can be stochastic, but then it will be perfectly correlated with the market price because we consider only one source of uncertainty. In the latter case the problem falls squarely into the realm of decision analysis. Indeed, the standard risk-neutral option price will be zero, yet a risk-averse investor would still place a positive value on the decision opportunity. Assumptions about the structure of investment costs turn out to be crucial for the ranking of alternative opportunities and for this ranking's sensitivity analysis to the investor's views on market prices and cash flows.

Standard risk-neutral real option prices are captured within our approach on assuming that the investment cost is predetermined, the forward asset price follows a GBM with total return equal to the risk-free rate, the investor applies a linear utility (equivalently, he has zero risk aversion, whatever his wealth) and the investment decision may be taken at any point in time. More generally, in an incomplete market the optimal decision path and the corresponding option value depend not only on the decision maker's risk preferences, but also on his subjective views about (a) the price he would pay or receive for the investment in future, and (b) the cash flows arising from owning the investment, which may be positive (e.g. rents from a buy-to-let apartment block or from a commercial office), negative (e.g. construction costs in an on-going development) or zero (for a self-occupied residential property).

Our general framework allows real option prices based on risk-neutral valuation to be compared with the values attributed by risk-averse investors in incomplete markets. We also answer several questions that have not previously been addressed in the literature, e.g. what is the effect of the investor's risk preferences (and especially, how his risk tolerance changes with wealth) on the ranking of different opportunities? How is this ranking influenced by the structure of the cash flows, or by the cost of the investment relative to the initial wealth of the investor? If the fixed-strike assumption is unlikely to be valid in practice, how does this assumption influence the option value? And how does the value change with the frequency of decision opportunities?

We proceed follows: Section 2 places our work in the context of the real options literature; Section 3 describes the model; Section 4 analyses the relationship between risk-neutral real option prices and different risk-averse option values and explains how these values are influenced by the frequency of decision opportunities, the relative cost of the investment, and the subjective views on expected return and risk. Section 5 analyses several real estate real options including buy-to-let and land development. We explain how the ranking of alternative investment opportunities depends on the structure of the cash flows, the investor's risk preferences – especially how his risk tolerance changes with wealth – and his views on market prices, including mean-reverting and 'boom-bust' price scenarios as well as standard GBM. Throughout, the standard risk-neutral real option price is computed as a special case, for comparison. Section 6 summarises and concludes.

2 Relevance to the Literature

Most of the literature on real options focuses on opportunities to enter a tradable contract with predetermined strike on an asset traded in a complete market (i.e. the real option's pay-off distribution can be replicated using tradable assets, so that all risks are hedgeable) and where the decision opportunity can be exercised continuously at any time over an infinite horizon. In this setting the option has the same value to all investors (Harrison and Kreps, 1981) and thus can be valued as if the investor is risk neutral, just like a financial option.

This strand of the literature regards a real option as a tradable contract, usually with a fixed or predetermined strike and a finite horizon.[1] As a result of applying the RNV principle there is a unique market price for the real option that is positively related to the volatility of the underlying asset.

In practice many decision opportunities encompassed by Myer's original definition are not standardized, tradable securities and their risks are only partially hedgeable, if at all. For instance, if an oil exploration company must decide whether to drill in location A or location B, its views about the benefits of drilling in each location will depend on their subjective beliefs about the market prices of oil in the future, as well as their risk preferences. And when a pharmaceutical company decides which drug to research and develop, both the research costs and the subsequent profits tend to be positively correlated with each drug's potential market price. In this incomplete market setting there is no unique value for a real option; it will be specific to the investor, depending on his subjective views about the stochastic costs and benefits, the form of his utility function and how his risk tolerance evolves with wealth. Importantly, and by contrast with the risk-neutral approach, a risk-averse investor's valuation of a real option may *de*crease as the underlying asset volatility increases, i.e. as the riskiness of the investment increases.

Closed-form, continuous-time, deterministic-strike real option values in incomplete or partially complete markets have been considered for decision makers with an exponential utility by Henderson (2002, 2007), Miao and Wang (2007) and Grasselli (2011). When the underlying asset is correlated with a market price Henderson (2007) applies the closed-form solution derived by Henderson (2002) for the exponential option value and the investment threshold, showing that market incompleteness results in earlier exercise and a lower real option value. Henderson and Hobson (2002) employ a power utility function in the Merton (1969) investment model, proposing an approximate closed-form optimal reservation value for the option when the investment cost is small relative to initial wealth. They show that the exponential and power utility option values behave very differently as a function of risk aversion as it tends to zero, due to boundary constraints on the power utility value. Evans et al. (2008) compare power and logarithmic utility values in a similar setting. Most of these papers derive the optimal investment threshold and the indifference value for a finite-horizon, continuous choice, deterministic-strike investment opportunity using a two-factor GBM framework in which the value of the project to the investor is stochastic and possibly

[1] See Triantis and Hodder (1990), Capozza and Sick (1991), Trigeorgis (1993), Panayi and Trigeorgis (1998), Benaroch and Kauffman (2000), Boer (2000), Yeo and Qiu (2002), Shackleton and Wojakowski (2007) and many other. Some papers assume an infinite horizon – see Kogut (1991), Grenadier (1996), Smith and McCardle (1998) and Patel et al. (2005) for a review. Other studies include a stochastic strike, including McDonald and Siegel (1986), Quigg (1993) and Bowman and Moskowitz (2001).

correlated with the price of a liquidly traded asset that may be used to hedge the investment risk. Both price processes are discounted to present value terms, so that a univariate time 0 utility function may be applied to maximize the expected utility of the maturity pay-off. Our work uses a similar discounted value process, but since we suppose from the outset that no risks are hedgeable by traded securities we utilize only a one-factor framework.

Like us, Grasselli (2011) considers the case where none of the risks of the project are hedgeable. Importantly, he proves that the time-flexibility of the opportunity to invest still carries an option value for a risk-averse investor, so that the paradigm of real options can be applied to value a private investment decision. Employing an exponential utility he proves analytically that the real option value converges to zero as risk tolerance decreases. Our numerical results demonstrate that this is also the case with other HARA utilities, where the real option value erodes at a rate that depends on the cost of the investment relative to the investor's wealth.

Kasanen and Trigeorgis (1995) propose a "market integrated utility" to value real options that are available to public companies, under the rationale that undertaking the underlying asset would increase the market value of the company. They calibrate various HARA utilities to market data and use this to value other, non-tradable assets of the company. By contrast, we employ an individual utility for a private company with purely subjective views on the evolution of the underlying asset price.

Some early work on real-option problems employed a decision-tree approach, but used a constant, risk-adjusted discount rate in place of a utility.[2] This led Copeland et al. (1990) to dismiss the decision-tree approach as being less accurate than option pricing methods, thus directing the main-stream of real options research towards risk-neutral American-type option valuation. Nevertheless, Trigeorgis (1996), Smit (1996), Brandão and Dyer (2005), Brandão et al. (2005, 2008) and Smith (2005) have all used a decision-tree approach, but these papers rely on the assumption of market-priced risk and thereby adopt a RNV framework.[3] The Integrated Valuation Procedure (IVP) introduced by Smith and Nau (1995) and extended by Smith and McCardle (1998) considers risk-averse decision makers endowed with a multivariate utility function defined as a sum of time-homogenous exponential utility functions of future cash flows, applying backward induction to derive the certain equivalent (CE) value of these utilities. The IVP approach has much in common with our own, but is only valid for an exponential utility and when cash flows at different times are independent, because only

[2]See Mason and Merton (1985), Trigeorgis and Mason (1987), Copeland et al. (1990) and Copeland and Antikarov (2001).

[3]The real estate real options that have been considered in this literature include: the option to defer a land development, e.g. Brandão and Dyer (2005) and Brandão et al. (2005), the option to divest, e.g. Brandão et al. (2008) and Smith (2005), and the option to abandon, e.g. Smit (1996).

exponential utilities have the unique property that their CE is additive over independent random variables. Also, although economic analysis is commonly based on inter-temporal consumption, it is standard in finance to base the utility of decisions on final wealth, with future values discounted to present value terms. This discounting is an important step in our framework because it greatly simplifies the decision analysis and allows the backward induction step to be defined on the expected utility relative to any univariate utility function, rather than on CE values based on an additively separable multivariate utility, as in the IVP. Additionally, since the resolution of our decision problem is numerical and not analytical, we have the freedom to employ any type of utility function we choose.

3 The Model

The mathematical framework may be summarized as follows: we assume investment risks are un-hedgeable, so the market for the underlying asset is incomplete; its forward market price has a measure that is subjective to the decision maker, with the risk-neutral measure being a special case; investment costs may be predetermined or, if stochastic, will be perfectly correlated with the market price; decision opportunities are discrete and are modeled using a binomial price tree with decision nodes placed at every k steps; the decision horizon T is finite; and the consequence of the decision is valued at some finite investment horizon, $T' > T$. The decision maker holds subjective views not only about the evolution of the market price for $0 \leq t \leq T'$ but also about the stream of cash flows (if any) that would be realised if he enters the investment. In most applications cash flows would reflect the individual management style of the decision maker, e.g. aggressive, expansive, recessive etc. Additionally, the decision maker is characterised by his initial wealth, w_0 which represents the current net worth of all his assets, and a HARA utility function $U(w)$ which reflects his risk tolerance λ and how this changes with wealth.

3.1 Market Prices and Cash Flows

All future market prices and cash flows are expressed in time 0 terms by discounting at the decision-maker's borrowing rate r.[4] Thus the investor borrows funds at rate r to buy the property, rather than financing the cost from his initial wealth, which we suppose is not

[4]This rate depends on the business risk of the project as perceived by the financer, not as perceived by the decision maker. It may also depend on the decision-maker's credit rating but this assumption is not common in the real options literature. The borrowing rate may be regarded as a cost of capital but it should not be risk-adjusted because we shall model the investor's utility values of the outcomes explicitly.

available for property investment.[5] We suppose that r is a constant, risk-free rate.

First we assume the market price of the underlying asset follows a GBM, so that the discounted forward market price p_t evolves over time according to the process:

$$\frac{dp_t}{p_t} = (\mu - r)dt + \sigma dW_t, \quad \text{for} \quad 0 < t \leq T', \tag{1}$$

where μ and σ are the decision-maker's subjective drift and volatility associated with p_t and W_t is a Wiener process. Then p_t has a lognormal distribution, $p_t \sim \log \mathrm{N}\left(\left(\mu - r\right)t, \sigma^2 t\right)$.

It is convenient to use a binomial tree discretisation of (1) in which the price can move up or down by factors u and d, so that $p_{t+1} = p_t u$ with probability π and otherwise $p_{t+1} = p_t d$. No less than eleven different binomial parameterisations for GBM are reviewed by Chance (2008). Smith (2005), Brandão and Dyer (2005), Brandão et al. (2005, 2008), Smit and Ankum (1993) and others employ the 'CRR' parameterization of Cox et al. (1979). However, the Jarrow and Rudd (1982) parameterisation, which is commonly used by option traders, is more stable for low levels of volatility and when there are only a few steps in the tree. Thus we set

$$m = \left[\mu - r - 0.5\sigma^2\right]\Delta t, \quad u = e^{m+\sigma\sqrt{\Delta t}}, \quad d = e^{m-\sigma\sqrt{\Delta t}} \quad \text{and} \quad \pi = 0.5. \tag{2}$$

For the applications to real estate real options we also consider a modification of (1) that represents a regime-dependent process, which trends upward with a low volatility for a sustained period and downward with a high volatility for another sustained period, thus replicating booms and busts in the market price of property. Thus we set:

$$\frac{dp_t}{p_t} = \begin{cases} (\mu_1 - r)dt + \sigma_1 dW_t, & \text{for} \quad 0 < t \leq T_1, \\ (\mu_2 - r)dt + \sigma_2 dW_t, & \text{for} \quad T_1 < t \leq T'. \end{cases} \tag{3}$$

An alternative modification of GBM arises when the decision maker believes the market price of property will mean-revert over a relatively short time horizon, rather than following upward/downward trends. To replicate this type of property price scenario we utilize a simple mechanism whereby the expected return would decrease following a price increase but increase following a price fall, as in the Ornstein-Uhlenbeck process:

$$d\ln p_t = -\kappa \ln\left(\frac{p_t}{\bar{p}}\right)dt + \sigma dW_t \tag{4}$$

[5] To avoid additional complexity we do not consider that investments could be financed from wealth, even though this would be rational if wealth is liquid and r is greater than the return on wealth, $\tilde{r}$.

where κ denotes the rate of mean reversion to a long-term price level $\bar{p}$. Following Nelson and Ramaswamy (1990) (NR) we employ the following binomial tree parameterisation for the discretised Ornstein-Uhlenbeck process:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = u^{-1} \tag{5a}$$

$$\pi_{\mathbf{s}(t)} = \begin{cases} 1, & 0.5 + \nu_{\mathbf{s}(t)}\sqrt{\Delta t}/2\sigma > 1 \\ 0.5 + \nu_{\mathbf{s}(t)}\sqrt{\Delta t}/2\sigma, & 0 \leq 0.5 + \nu_{\mathbf{s}(t)}\sqrt{\Delta t}/2\sigma \leq 1 \\ 0, & 0.5 + \nu_{\mathbf{s}(t)}\sqrt{\Delta t}/2\sigma < 0 \end{cases} \tag{5b}$$

where

$$\nu_{\mathbf{s}(t)} = -\kappa \ln\left(\frac{p_{\mathbf{s}(t)u}}{\bar{p}}\right) \tag{5c}$$

is the local drift of the log price process, which decreases as κ increases. The corresponding price process thus has local drift:

$$\mu_{\mathbf{s}(t)} = -\kappa \ln\left(\frac{p_{\mathbf{s}(t)u}}{\bar{p}}\right) + 0.5\sigma^2 + r \tag{5d}$$

Note that when $\kappa = 0$ there is a constant transition probability of 0.5 and the NR parameterisation is equivalent to the parameterisation (2) with $m = 0$.

The cash flows, if any, may depend on the market price of the asset as, for instance, in rents from a property. We suppose that he receives the cash flow if he divests in the project at time t but does not receive it if he invests in the project at time t. To define the cash flows let $\mathbf{s}(t)$ denote the state of the market price at time t, i.e. a path of the market price from time 0 to time t. In the binomial tree framework $\mathbf{s}(t)$ may be written as a string of u's and d's with t elements, e.g. uud for $t = 3$. Now $\mathrm{CF}_{\mathbf{s}(t)}$ denotes the cash flow when the market price is in state $\mathbf{s}(t)$ at time t. Regarding cash flows as dividends we call the price excluding all cash flows before and at time t the 'ex-dividend' price, denoted $p^-_{\mathbf{s}(t)}$. At the time of a cash flow $\mathrm{CF}_{\mathbf{s}(t)}$ the market price follows a path which jumps from $p^+_{\mathbf{s}(t)} = p^-_{\mathbf{s}(t)} + \mathrm{CF}_{\mathbf{s}(t)}$ to $p^-_{\mathbf{s}(t)}$.[6]

The dividend yield, also called dividend pay-out ratio, is defined as

$$\delta_{\mathbf{s}(t)} = \frac{p^+_{\mathbf{s}(t)} - p^-_{\mathbf{s}(t)}}{p^+_{\mathbf{s}(t)}}. \tag{6}$$

[6] The subscript $\mathbf{s}(t)$ denotes a particular realisation of the random variable that carries the subscript t, e.g. p^+_{uud} and p^-_{uud} are the left and right limits of a realisation of p_t when $t = 3$. From henceforth we use either the subscript $\mathbf{s}(t)$ or the subscript t, whichever is appropriate in the context of the price or cash flow. Also, to be clear, p^-_t is a limit of p_t from the right, not a limit from the left as is the usual notation in the literature for jump processes.

Clearly, if the cash flow is not state dependent, then the dividend yield must be state dependent; if the dividend yield is not state dependent, then the cash flow must be state dependent; in the general case both will be state dependent. The most basic model assumes that dividend yields are deterministic and time but not state dependent, in which case we use the simpler notation δ_t; in this case note that the cash flows will be both time and state dependent.[7]

3.2 Costs and Benefits of Investment and Divestment

The investment cost at time t, in time 0 terms, is given by:

$$K_{\mathbf{s}(t)} = \alpha K + (1-\alpha)p^-_{\mathbf{s}(t)}, \quad 0 \leq \alpha \leq 1, \tag{7}$$

where K is a constant in time 0 terms. When $\alpha = 1$ we have a standard real option with a predetermined strike K, such as might be employed for oil exploration decisions. When $\alpha = 0$ we have a variable cost at the market price $p^-_{\mathbf{s}(t)}$, such as might be employed for real estate or merger and acquisition options. The intermediate case, with $0 < \alpha < 1$ has an investment cost with both fixed and variable components, as may be the case with research and development decisions.

We suppose that initial wealth w_0 earns a constant, risk-free lending rate $\tilde{r}$, as do any cash flows paid out that are not re-invested in the property (or similar investment). Any cash paid into the property (e.g. a land development cost) is financed at the borrowing rate r. The financial benefit to the decision-maker on investing at time t is the sum of any cash flows paid out and not re-invested plus the terminal market price of the project. The cost of entering the project at time t, in time 0 terms, is K_t, given by (7). Thus the wealth of the investor at time T', in time 0 terms, following investment at time t is

$$w^I_{t,T'} = e^{(\tilde{r}-r)T'}w_0 + \sum_{s=t+1}^{T'} e^{(\tilde{r}-r)(T'-s)}\mathrm{CF}_s + p^-_{T'} - K_t. \tag{8}$$

Some investments pay no cash flows, or any cash flows paid out are re-invested in the project. Then the financial benefit of investing at time t, in time 0 terms, is simply the cum-dividend price $\widetilde{p}_{t,T'}$ of the project accruing from time t. If a decision to invest is made at time t, with $0 \leq t \leq T$, then $\widetilde{p}_{t,t} = p_t$ but the evolution of $\widetilde{p}_{t,s}$ for $t < s \leq T'$ differs to that of p_s because $\widetilde{p}_{t,s}$ will gradually accumulate all future cash flows from time t onwards. In this case, when the decision maker chooses to invest at time t his wealth at time T' in time

[7] The state dependence of cash flows induces an autocorrelation in them because the market price is autocorrelated. For this reason, defining an additively separable multivariate utility over future cash flows as in Smith and Nau (1995) and Smith and McCardle (1998) is problematic.

0 terms is

$$\widetilde{w}^{I}_{t,T'} = e^{(\tilde{r}-r)T'} w_0 + \widetilde{p}_{t,T'} - K_t. \tag{9}$$

Note that if $\tilde{r} = r$ then also $\sum_{s=t+1}^{T'} e^{(\tilde{r}-r)(T'-s)}\mathrm{CF}_s + p^{-}_{T'} = \tilde{p}_{t,T'}$ in (8).

Similarly, if the decision maker already owns the investment at time 0 and chooses to sell it at time t, the time 0 value of his wealth at time T' is

$$w^{S}_{t,T'} = e^{(\tilde{r}-r)T'} w_0 + \sum_{s=1}^{t-1} e^{(\tilde{r}-r)(T'-s)}\mathrm{CF}_s + p^{+}_t - p_0. \tag{10}$$

Note that p_0 is subtracted here because we assume the investor has borrowed funds to invest in the property. Alternatively, if there are no cash flows, or they are re-invested,

$$\widetilde{w}^{S}_{t,T'} = e^{(\tilde{r}-r)T'} w_0 + \widetilde{p}_{0,t} - p_0. \tag{11}$$

The wealth $w^{D}_{t,T'}$ resulting from a defer decision at time t depends on whether he invests later on. This must therefore be computed using backward induction as described in the next sub-section.

3.3 Optimal Decisions and Real Option Value

As in any decision problem, we shall compare the expected utility of the outcomes resulting from investment with the utility of a base-case alternative, which in this case is to do nothing.[8] For brevity, we describe the backward induction step only for the decision to invest – it is similar for the decision to divest, but replace I with S (for an existing investor that sells the property) and D with R (for an existing investor that remains invested).

The option to invest at time t has time 0 utility value $U^{I}_{t,T'} = U\left(w^{I}_{t,T'}\right)$, but since $w^{I}_{t,T'}$ is random so is $U^{I}_{t,T'}$, and we use the expected utility

$$\mathrm{E}\left[U^{I}_{\mathbf{s}(t),T'}\right] = \mathrm{E}\left[U\left(w^{I}_{t,T'}\right)\right],$$

as a point estimate. Then, given a specific decision node at time t, say when the market is in state $\mathbf{s}(t)$, the potential investor chooses to invest if and only if

$$\mathrm{E}\left[U^{I}_{\mathbf{s}(t),T'}\right] > \mathrm{E}\left[U^{D}_{\mathbf{s}(t),T'}\right],$$

[8] So, for the divestment decision we compare the expected utility of the outcomes resulting from remaining invested with the utility of the alternative, to divest.

and we set

$$\mathrm{E}\left[U_{\mathbf{s}(t),T'}\right] = \max\left\{\mathrm{E}\left[U^I_{\mathbf{s}(t),T'}\right], \mathrm{E}\left[U^D_{\mathbf{s}(t),T'}\right]\right\}. \tag{12}$$

Since there are no further decisions following a decision to invest, $\mathrm{E}\left[U^I_{\mathbf{s}(t),T'}\right]$ can be evaluated directly, using the utility of the terminal wealth values obtainable from state $\mathbf{s}(t)$ and their associated probabilities. However, $\mathrm{E}\left[U^D_{\mathbf{s}(t),T'}\right]$ depends on whether it is optimal to invest or defer at the decision nodes at time $t+1$. Thus, the expected utilities at each decision node must be computed via backward induction.

First we evaluate (12) at the last decision nodes in the tree, which are at the time T that option expires. These nodes are available only if the investor has deferred at every node up to this point. We associate each ultimate decision node with the maximum value (12) and select the corresponding optimal action, I or D. Now select a penultimate decision node; say it is at time $T - k\Delta t$. If we use a recombining binomial tree to model the market price evolution, it has 2^k successor decision nodes at time T.[9] Each market state $\mathbf{s}(T - k\Delta t)$ has an associated decision node. Each one of its successor nodes is at a market state $\mathbf{s}^*(T)$ that is attainable from state $\mathbf{s}(T - k\Delta t)$, and has an associated probability $\pi_{\mathbf{s}^*(T)}$ determined by the state transition probability of 0.5, given that we employ the parameterisation (2).[10] Using the expected utility associated with each attainable successor node, and their associated probabilities, we compute the expected utility of the decision to defer at time $T - k\Delta t$. More generally, assuming decision nodes occur at regular time intervals, the backward induction step is:

$$\mathrm{E}\left[U^D_{\mathbf{s}(t-k\Delta t),T'}\right] = \sum_{\mathbf{s}^*(t)} \pi_{\mathbf{s}^*(t)} \mathrm{E}\left[U_{\mathbf{s}^*(t),T'}\right], \quad t = k\Delta t, 2k\Delta t, \ldots, T - k\Delta t, T. \tag{13}$$

At each decision node we compute (13) and associate the node with the optimal action and its corresponding maximum expected utility. We repeat the backward induction until we arrive at a single expected utility value associated with the node at time 0. Finally, the option value is the certain equivalent (CE) of this expected utility, less the initial wealth w_0. By definition, $\mathrm{CE}(w) = U^{-1}(\mathrm{E}[U(w)])$ for any monotonic increasing utility U.

[9] The recombining assumption simplifies the computation of expected utilities at the backward induction step. However, we do not require that the binomial tree is recombining so the number of decision nodes could proliferate as we advance through the tree. Note that the state price tree will recombine if cash flows are determined by a time-varying but not state-varying dividend yield.

[10] So if the tree recombines these probabilities are $0.5^k, k0.5^k, k!/(2!(k-2)!)0.5^k, ..., k0.5^k, 0.5^k$ under the JR parameterization (2). If the CRR parameterisation is employed instead the transition probabilities for a recombining tree would be more general binomial probabilities.

3.4 Risk Preferences

Denote by $U : \mathcal{R} \rightarrow \mathcal{R}$ the decision-maker's utility function. Previous research on decision analysis of real options, reviewed in Section 2, employs either risk-neutrality or an exponential utility function, which may be written in the form

$$U(w) = -\lambda \exp\left(-\frac{w}{\lambda}\right), \tag{14}$$

where w denotes the terminal (time T') wealth of the decision maker, expressed in time 0 terms and $\lambda > 0$ denotes his risk tolerance and $\gamma = \lambda^{-1}$ is his risk aversion. Note that w is a random variable taking values determined by the decision-maker's (subjective) views on the evolution of the market price and the decisions he takes before time T'. Under (14) we have

$$\mathrm{CE}(w) = -\lambda \log\left(-\frac{\mathrm{E}[U(w)]}{\lambda}\right). \tag{15}$$

The exponential utility function has very special properties that make it particularly tractable (see Davis et al., 2006, Chapter 6). In fact, all the properties listed below are unique to exponential utilities except the last, which is shared by other utility functions in the HARA class.

1. The exponential function (14) is the only utility with a CE that is independent of the decision-maker's initial wealth, w_0. In other words, adding a constant to w results in only in an affine transformation which does not change the form of the utility. This is also known as the 'delta' property. It implies that we obtain the same option value, and the same optimal decisions, whether w denotes the P&L or net wealth;

2. The Arrow-Pratt coefficient of risk aversion $-U''(w)/U'(w) = \gamma$. Thus, the exponential utility (14) represents decision makers with constant absolute risk aversion (CARA) and λ in (14) has the approximate interpretation of being the maximum dollar amount X that one would be willing accept in a 50:50 gamble of winning X and losing $X/2$;

3. The exponential is the only utility function with an additive CE over independent risks. When $x_t \sim$NID(μ, σ^2) and $w_T = x_1 + \ldots + x_T$ then $\mathrm{CE}(w_T) = \mu T - (2\lambda)^{-1}\sigma^2 T$.

4. Exponential utilities have the following time-homogeneity property. Suppose $U_t : R \rightarrow R$ on $t = 0, 1, ..., T'$, is defined as

$$U_t(w_t) = -\lambda_t \exp\left(-\frac{w_t}{\lambda_t}\right), \tag{16}$$

with risk tolerance $\lambda_t = e^{rt}\lambda$. Set $w_t^0 = e^{-rt}w_t$. Then:

$$U_t(w_t) = -\lambda_t \exp\left(-\frac{w_t}{\lambda_t}\right) = -e^{rt}\lambda \exp\left(-\frac{w_t^0}{\lambda}\right) = e^{rt}U(w_t^0),$$

where $U : R \to R$ is a time-invariant exponential utility function as in (14) defined on any future value of wealth discounted to time 0.

Properties 1 and 2 are very restricting. CARA implies that decision makers leave unchanged the dollar amount allocated to a risky investment when their initial wealth changes. The delta property means that the initial wealth of the decision maker, and therefore also its size relative to the price of the asset, has no influence on the exponential utility option value. Property 3 implies that when cash flows are normally and independently distributed (NID) the decision-maker's risk premium for the sum of cash flows at time t is $(2\lambda)^{-1}\sigma^2 T$. The risk premium scales with time at rate $(2\lambda)^{-1}\sigma^2$, which is the risk-adjustment term that is commonly applied to discounted cash flow models and was also applied in the influential paper by Copeland et al. (1990). Thus, risk-adjusted discount rates implicitly assume an exponential utility over NID uncertainties and when the decision-maker's utility is explicitly modelled there is no need to additionally risk-adjust discount rates.

The last property makes an exponential utility (and indeed any HARA utility) particularly easy to employ in decision-tree analysis. In particular, on assuming the risk tolerance is time-varying and grows exponentially at the same rate as the discount rate, we can compute utilities (and their expectations and certain equivalents) to value any future uncertainty by using the constant utility function (14) applied to present values. This is much easier than discounting the expected values of time-varying utility functions applied to time t values at every step in the backward induction. However, our framework is not constrained to exponential preferences over NID uncertainties. In fact, such a case would lead to a solution where the same decision (to invest, or to defer) would be reached at every decision node in the tree, because the uncertainties faced at any node are just a scaled version of the uncertainties at any other node.

However, the CARA property of the exponential utility is often criticised because it would not apply to investors that would change the dollar amount allocated to risky investments when their wealth changes. For this reason we now consider a selection of other utility functions from the HARA class for which both absolute and relative risk aversion can increase with wealth.

The displaced logarithmic utility may be written

$$U(w) = \lambda w_0 \log\left(1 + \frac{w - w_0}{\lambda w_0}\right), \quad \text{for } w > (1-\lambda)w_0, \tag{17}$$

where w includes initial wealth w_0, so $w - w_0$ is the P&L from the investment. Note that (17) is a standardized from of logarithmic utility for which $U(w_0) = 0$, $U'(w_0) = 1$ and the asymptote to $-\infty$ is at $(1-\lambda)w_0$. Also, $-U'(w)/U''(w) = \lambda w_0 + w - w_0$, so at $w = w_0$ the local coefficient of absolute risk tolerance is λw_0 and λ is the local coefficient of *relative* risk tolerance.[11] When $\lambda = 1$, local relative risk aversion is 1, independent of wealth, so in this case the utility (17) is said to have the constant relative risk aversion (CRRA) property.

The power utility is best written in terms of risk aversion $\gamma = \lambda^{-1}$, as

$$U(w) = -\gamma|1-\gamma|^{-1}\left(\frac{w}{w_0}\right)^{1-\gamma}, \quad \text{for } w/w_0 > 0 \text{ if } \gamma > 1. \tag{18}$$

Since $-U'(w)/U''(w) = \lambda w$, λ is the local coefficient of relative risk tolerance. It is a constant, so the power utility has the CRRA property. Absolute risk tolerance increases linearly w, but for (18) the sensitivity of local risk tolerance to changes in w is λ, whereas it is 1 for (17). Thus, when $\lambda < 1$, local risk tolerance for (17) is greater than (less than) it is for (18) when $w > w_0$ ($w < w_0$). The opposite is the case when $\lambda > 1$.

Exponential, logarithmic and power utilities are all special cases of the general HARA utility, which has two parameters determining the local coefficient of absolute risk tolerance and its sensitivity to wealth. That is, HARA utility functions have a risk tolerance λ that increases linearly with wealth at the rate η, and are defined as:

$$U(w) = -[1 + \frac{\eta}{\lambda w_0}(w - w_0)]^{1-\eta^{-1}}(1-\eta)^{-1}, \quad \text{for } w > (1-\eta^{-1}\lambda)w_0. \tag{19}$$

When $\eta = 0$ we have the exponential utility, $\eta = 1$ corresponds to the displaced logarithmic utility, $\eta = 0.5$ gives the hyperbolic utility and $\eta = \lambda$ gives the power utility.

4 Numerical Results

Recall that RNV principal yields a unique price for an option, and this reference price implies that the option could be tradable on a secondary market. By contrast, the option value for a

[11] Relative risk tolerance is expressed as a percentage of wealth, not in dollar terms. So if, say, $\lambda = 0.4$ the decision maker is willing to take a gamble with approximately equal probability of winning 40% or losing 20% of his wealth, but he would not bet on a 50:50 chance (approximately) of winning x% or losing x%/2 for any $x > 0.4$.

risk-averse investor in an incomplete market is purely subjective. It represents the net present value that, if received with certainty, would give him the same utility value as the expected utility of the uncertain investment.[12] Such values merely enable the investor to rank alterative investment opportunities.

With the aim of addressing some of the questions raised in the introduction, the results in this section compute the values of several real options under different assumptions for the investor's utility but always assuming his views correspond to the GBM price process assumption (1). We analyse the relationship between the RNV option price and the more general, subjective option values that apply under the incomplete market assumption. We illustrate how the subjective value changes with the scheduling of decision opportunities, and with the cost of the investment relative to the companies' net asset value (or initial wealth). We explain how the decision maker's risk preferences influence his valuation, and especially how the sensitivity of his risk tolerance to wealth can alter his ranking of different opportunities. And we show how crucial the assumption about predetermined or stochastic investment costs really is, and how it influences the sensitivity of real option values to the investor's views about expected return and risk.

First we present a simple example to help fix ideas, where the decision maker has an exponential utility and the transacted price is the market price of the asset, not a predetermined strike. The investment decision thus provides a concrete example of the zero correlation case considered by Grasselli (2011), where the opportunity to invest still carries a positive value. Indeed, the equivalent divest decision also has a positive value. From henceforth, unless otherwise stated, we set $\tilde{r} = r$; no additional insights to the questions that we pose can be provided by using differential lending and borrowing rates. Also, we set the current price of the asset to be $p_0 = \$1$m, without loss of generality.

4.1 Real Options at Market Price: Incomplete Market, Exponential Utility

Consider example of an option to purchase an asset that has no associated cash flows where decisions can be taken now, next year and the year after and the investment horizon is $T' = 3$ years. The current price of the asset is one million dollars and the decision maker believes this will evolve according to (1) with $\mu = 10\%$ and $\sigma = 20\%$ per annum. He has an exponential utility with risk tolerance 0.4, and the risk-free lending and borrowing rates are both 5%.[13] The decision tree is depicted in Figure 1. We set $\Delta t = k = 1$, $r = 5\%$,

[12] Just like financial option prices, the minimum subjective value for a real option is zero, representing that the investment would never be attractive whatever happens to its market price in future.

[13] To facilitate our later comparison between exponential and other types of utility values for real options, we represent λ as a proportion of initial wealth. Since the exponential utility is independent of initial wealth

$\mu = 10\%$, $\sigma = 20\%$, so using (2) we have $m = 0.03$, $u = 1.259$, $d = 0.844$ and $\pi = 0.5$. Decision nodes are marked by shaded blue boxes, with the alternatives I and D denoting invest and defer. Black numbers are discounted market prices, starting with $p_0 = 1$. Red numbers at terminal nodes are the purchase cost and, after the '=' sign, the P&L from investment in time 0 terms.[14] For instance, if the decision is to invest at $t = 2$ and the market price moves up every year, the present value of the profit equals the difference between 1.99 (the market price in year 3) and 1.58 (the cost after two up moves) i.e. 0.41. Numbers in blue at the terminal nodes are utility values, using (14). For instance, at the lowest nodes $-0.4 = U(0)$, $-0.528 = U(-0.11)$ and $-0.252 = U(0.18)$. Framed blue numbers are expected utilities, e.g. $-0.390 = 0.5 \times -0.252 + 0.5 \times -0.528$.

Consider the application of (12), first at the $t = 2$ decision nodes that follow a decision to defer at $t = 1$. The lowest of these four nodes takes the utility value $\max\{-0.390, -0.4\} = -0.390$ so the optimal decision is to invest, which is marked in bold. A similar argument applied to the three other decision nodes at $t = 2$ gives an optimal decision to defer in each case. Next consider the two decision nodes at $t = 1$ and again compare the expected utility from investment to that of deferring. Investment at $t = 1$ leads directly to three possible terminal values for P&L. For instance, following an up move at time 0 we reach the upper of the two decision nodes. The attainable terminal market prices are $1.99, 1.34$ and 0.9 with associated probabilities 0.25, 0.5 and 0.25 respectively, so the expected utility is -0.428. But in this case the expected utility from deferring is -0.4, which is greater, so the optimal decision is to defer and the node takes the utility value -0.4. A similar argument for the lower node yields the optimal decision to invest, with an expected utility of -0.388. Thus, applying (13), the expected utility from deferring at time 0 is -0.394. This is compared with the expected utility of -0.399 from investing at time 0, which can be calculated directly using the four possible values for terminal wealth at the top of the tree, having associated probabilities 0.125, 0.375, 0.375 and 0.125 respectively. Since $-0.394 > -0.399$, the optimal decision at time 0 is to defer, and the expected utility value at time 0 is -0.394. The certain equivalent of this is $-0.4 \log(0.394/0.4) = 5,853 \times 10^{-6}$, which is the net value of the real option to this decision maker, measured in the same units as the market price. Hence, the optimal decision is to defer the investment now and invest in the future only if the market

an alternative interpretation is that the decision maker has an initial wealth of $1m.

[14] Although this is not the case in general, with an exponential utility one can leave the initial wealth out of the entire analysis. So in Figure 1 and 2 the terminal price nodes show only the increment in wealth, i.e. profit and loss, P&L. Exactly the same option value would obtain on adding w_0 to all terminal price nodes and subtracting w_0 from the final CE at $t = 0$. Then the expected utility values in the tree would change, but the same decisions would be optimal at every node and we would arrive at the same real option value. Indeed, with an exponential utility we can add any constant we like to all terminal nodes in the tree, provided we subtract this constant from the final CE at time 0.

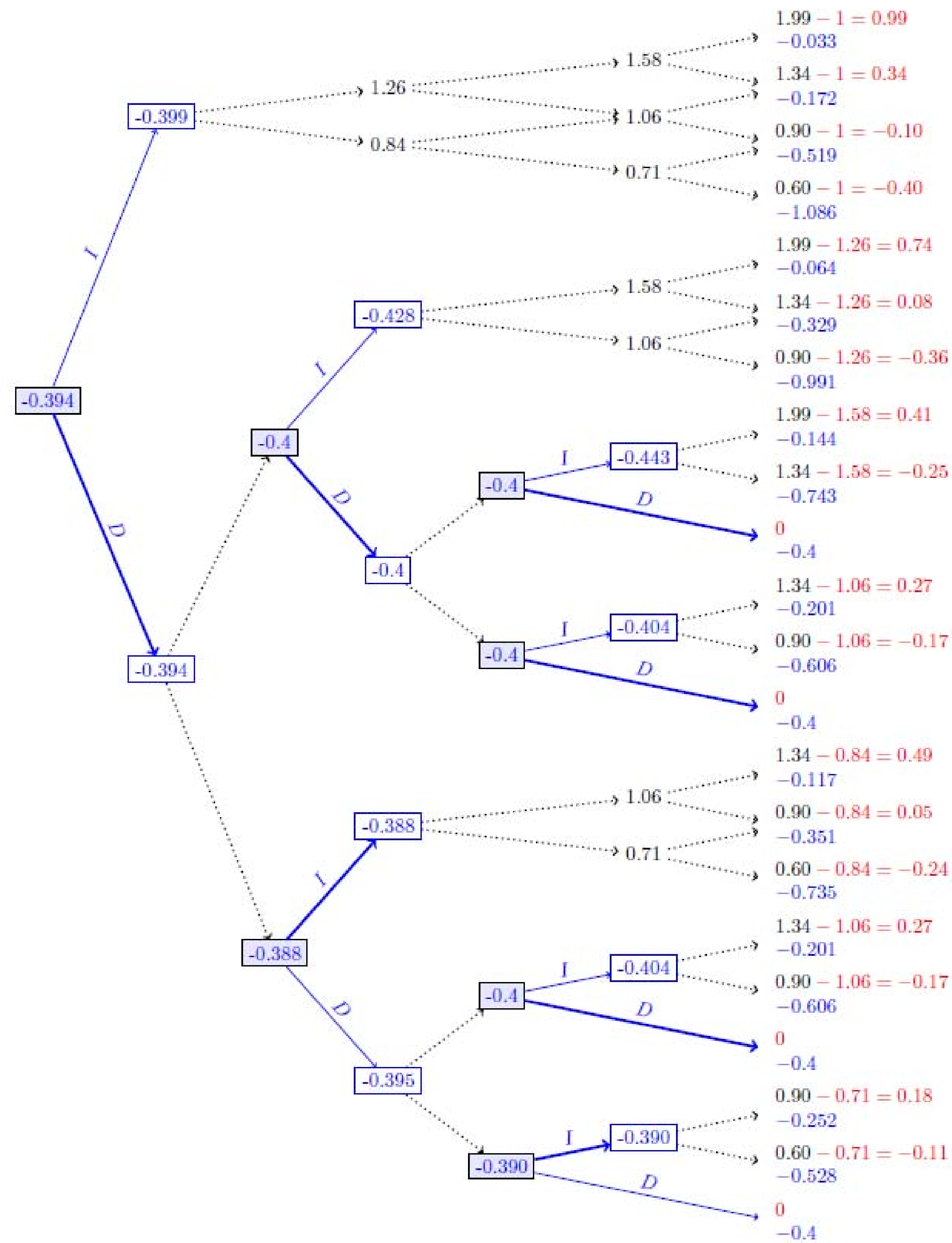


Figure 1: Illustration of the backward induction for the invest option, including the optimal decision at each node with I = 'invest' and D = 'defer', discounted market prices in black, expected utilities for backward induction in blue, costs and P&L in red. Decision opportunities every year for three years. $p_0 = 1$, $r = 5\%$, $\mu = 10\%$, $\sigma = 20\%$. Exponential utility with $\lambda = 0.4$.

price falls next year, and since the asset is currently valued at \$1m, the value of this decision opportunity is 5,853 after accounting for the cost of capital to purchase the property.

Now consider the equivalent problem from the perspective of the current owner of the asset.[15] We suppose this potential seller has the same views on market prices as the potential investor, and the same utility and borrowing/lending costs. Figure 2 depicts the decision tree for the divest option. The action S denotes 'sell the property' and the action R denotes 'remain invested'. To obtain the present value of the P&L arising from selling the asset at any time in future, we subtract its current value p_0 from the forward market price, assuming the asset is currently financed by borrowing funds.

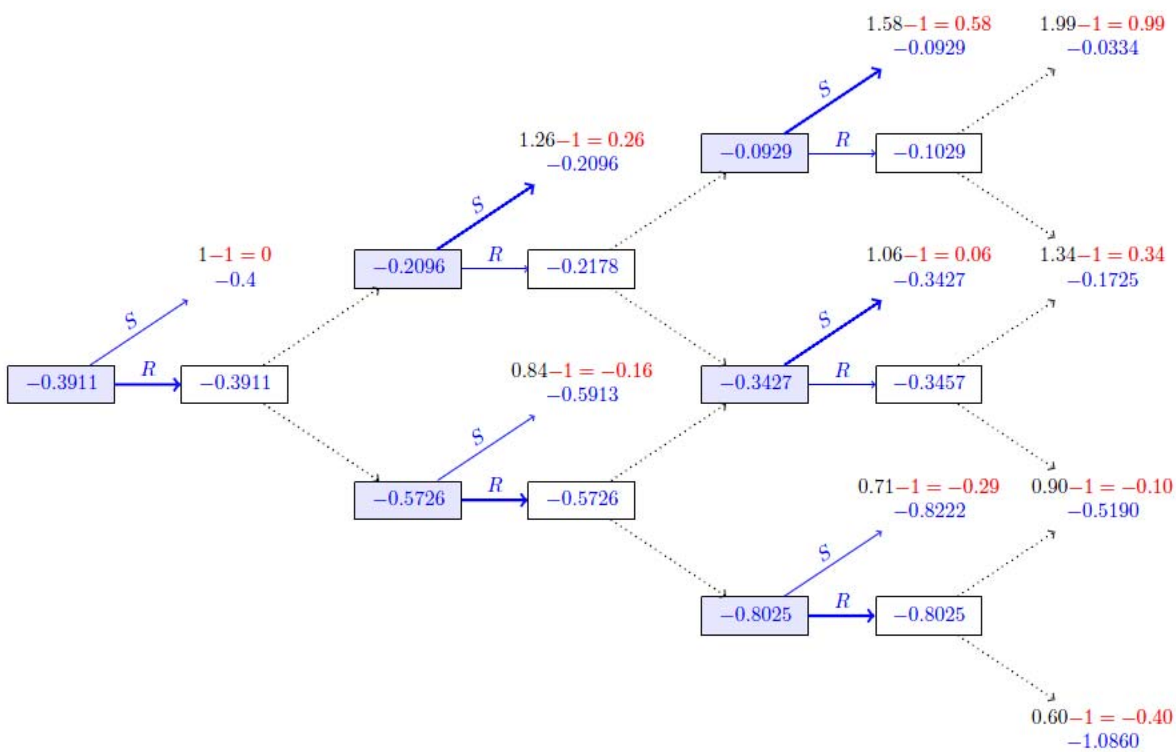


Figure 2: Illustration of the divest real option, including optimal decision at each node with S = 'sell' and R = 'remain', discounted market prices in black and expected utilities for backward induction in blue. Decision opportunities every year for three years. $r = 5\%$, $\mu = 10\%$, $\sigma = 20\%$. Exponential utility with $\lambda = 0.4$.

If the owner sells the property the market price at that time is realised and there is no future uncertainty. But if he remains the owner of the property, further decision nodes occur

[15]We use the term 'equivalent' to refer to the invest and divest options that are based on the same price and decision processes, for decision makers with identical utilities. Note that if it is optimal for the investor to exercise his option at $t = 0$ and for the divestor of the equivalent option never to exercise, their pay-off distributions become identical and hence the two options have the same value – see XXX for an example.

at future times when the market price is uncertain. If he does not sell the property at or before $t = 2$, the terminal market prices can take one of two values, given the price at the penultimate node. For instance, if the market price moves up at $t = 1$ and at $t = 2$, then at $t = 3$ it could be either 1.99 or 1.34, with equal probability. Now the computation of expected utilities and application of (12) and (13) proceed exactly as before. The expected utility at time 0 is -0.0321 and its CE is $-0.4\log(0.0321/0.4) = 9,030 \times 10^{-6}$. We conclude that this real option to divest has value \$9,030 to this particular owner, whose optimal decision is to keep the asset now but sell it if the price goes up next year.

The current price p_0 plays the role of a fixed cost to the divestor. This contrasts with the investment option, where the investor pays the market price of the asset at the time he chooses to invest, so his cost is not constant. As a result, when the parameters that determine the market price are changed the pay-off distribution of the divest option fluctuates at least as much, and often more than that of the equivalent invest option. Thus the divest option value is at least as sensitive as the equivalent invest option to changes in the parameters of the price process. We shall explore the sensitivity of real option values to changes in the decision maker's subjective views about expected return and risk in Section 4.6.

4.2 Investment Costs and the Risk-Neutral Valuation Approach

Again consider the example of Figure 1, but now allow the investment cost to take the general form (7) with $K = \$1\text{m}$. We compare the case $\alpha = 0$ considered above with the fixed-strike option where $\alpha = 1$, and with the case $\alpha = 0.5$. The investment threshold depends crucially on α: when $\alpha = 0$ (invest at market price) it may become optimal to invest but only if the price falls, as shown above. However, when $\alpha = 1$ so we have a fixed strike, at-the-money (ATM) option, it may become optimal to invest but only if the price rises because in this case the option value depends on the expected utility of an ATM call option pay-off.

Table 1 shows how the option value increases with both α and the investor's risk tolerance. The value of \$5,853 that was derived using Figure 1 is shown in bold. The real option values increases with both α and λ. As $\lambda \to \infty$ the option value converges to the risk-neutral (linear utility) value, and for the fixed-cost case $\alpha = 1$ we obtain the maximum value of \$224,333. This value is still contingent on the investor's personal views about market prices. In the case that the investment risks are perfectly hedgeable, as under the standard RNV assumptions, the option price is \$108,285. The RNV price is the same for all investors (independent of λ) and it can be derived in our framework using a linear utility and setting $\mu = r$ (in this case 5%).

Great care should be taken when making assumptions about investment costs. In some

α \ λ	0.2	0.4	0.6	0.8	1	∞
0	0	**5,853**	45,305	70,656	86,976	161,374
0.5	31,597	64,814	84,749	96,762	106,544	161,374
1	59,825	116,939	145,009	161,573	172,462	224,333

Table 1: Exponential utility with parameters $T' = 3$ years, $\Delta t = k = 1$, $K = p_0 = \$1$m, $r = \tilde{r} = 5\%$, $\mu = 10\%$, $\sigma = 20\%$.

applications – for instance, when a licence to drill for oil has been purchased and the decision concerns whether the market price of oil is sufficient to warrant exploration – the fixed-strike assumption would be applicable. However, in many applications the investment cost is at the market price prevailing at the time that the investment is made. The assumption about the investment cost – whether it is fixed (in time 0 terms) or stochastic (and perfectly correlated with the market price) – has a significant influence on the real option value. A fixed cost ($\alpha = 1$) may be regarded as the strike of an American call option, and the real option value is derived from the expected utility of a call option pay-off for which the upper part of the terminal wealth distribution above the strike matters. In contrast, the opposite extreme ($\alpha = 0$) focuses on the lower part of the terminal wealth distribution below the current price p_0. Although log returns are similar across the whole spectrum under the GBM assumption (1), P&L is in absolute terms and it is greater in the upper part of the distribution than in the lower part. For this reason the at-the-money ($K = \$1$m) fixed-strike assumption always yields a greater real option value than the invest-at-market-price assumption. However, this observation only holds under GBM views for market prices, see Section 5.1 for further details.

Our second conclusion is that the standard, RNV approach can produce very unreliable results when the assumption of a complete market and a fixed strike option are unwarranted. The subjective price could be very much less than the RNV price (especially when risk tolerance is low and the investment cost is at market price) or very much greater than the RNV price (especially when risk tolerance is high and the investment cost is fixed). Note that the RNV price is always zero when the opportunity to invest is at market price. This is because the linear utility function yields a CE equal to the expected value of terminal wealth, and this is w_0 since the discounted price is always a martingale under the risk-neutral measure. In such a case the value of an investment opportunity is always zero.[16] The relationship between the RNV option price and a real option's more general utility valuation is further explored in Section 4.5.

[16]The RNV price is based on a hedging argument that also requires no frictions, i.e. a perfect market, so that lending and borrowing costs are the same, i.e. $r = \tilde{r}$ as we have assumed.

4.3 Effect of Decision Frequency on Real Option Value

A real option value should not decrease when there is more flexibility to make decisions over the horizon of the option. If the decision never changes as a result of including more or less decision nodes, the option value will remain unchanged. Otherwise, the option value should increase as more decisions are allowed, since having fewer decisions places additional constraints on the opportunities available to the decision maker. Here we quantify the effect on the option value of increasing the number of decision opportunities. It is important that the trees are nested, i.e. no new decision nodes are inserted as their number decreases, because only in this way does reducing the number of nodes capture the effect of placing additional constraints on decision opportunities.

$\alpha = 0$: invest at market price

k \ λ	0.2	0.4	0.6	0.8	1	∞
12	109.5	1,132	3,712	8,197	14,480	645,167
6	142.2	1,344	4,309	9,212	16,089	645,167
3	163.9	1,471	4,618	9,803	17,022	645,167
1	176.9	1,553	4,828	10,185	17,624	645,167

$\alpha = 1$: fixed strike with present value \$1m

k \ λ	0.2	0.4	0.6	0.8	1	∞
12	49,385	115,086	174,731	223,201	263,614	881,419
6	71,365	144,638	204,078	252,243	292,632	908,333
3	86,062	157,289	214,375	263,019	303,834	919,322
1	93,115	166,450	225,568	273,619	313,888	926,058

Table 2: Effect of number of decisions on real option value. Exponential utility, for different λ and k. $p_0 = \$1m, T' = 5, \Delta t = 1/12, T = T' - k\Delta t, r = \tilde{r} = 5\%, \mu = 15\%, \sigma = 50\%$.

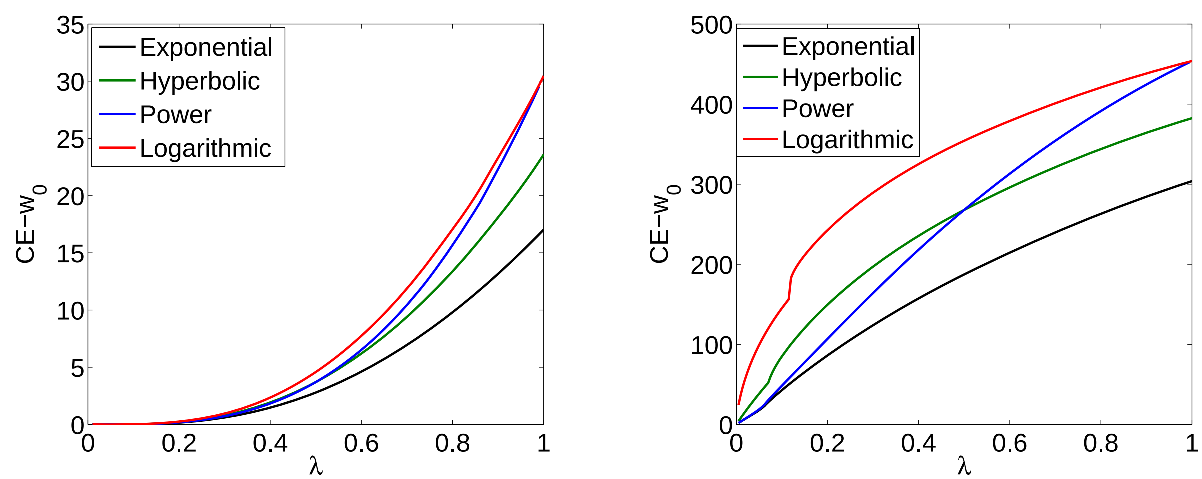
The results shown in Table 2 are based on an opportunity to invest in an asset that has no associated cash flows and has current market price \$1m. The investor has an exponential utility and the decision tree (not shown) is characterised by the parameter values: $T' = 5, \Delta t = 1/12, r = 5\%, \mu = 15\%, \sigma = 50\%$. Thus, there are 60 monthly steps in the binomial tree for the market price.

Now suppose that decision nodes occur every k steps, as in the backward induction algorithm (13), and that $T = T' - k\Delta t$. For instance, if $k = 12$ the decision nodes occur only once per year and the last decision is taken at the fourth year. So that the four decision trees are nested the values considered for k are $12, 6, 3$ and 1 representing decision opportunities once per year and once every 6 months, 3 months and 1 month.

The upper part of Table 2 reports the value of the option to invest at market price and the lower part reports the value of the option to invest at a fixed cost $K = \$1\text{m}$, for investors with different levels of risk tolerance λ. Again the option value increases very rapidly with λ. As expected, it also increases when more decision nodes occur in the tree, i.e. as k decreases, and the percentage increase in option value is greatest for investors with low λ, whereas the absolute increase in option value is greatest for high λ.

4.4 The HARA Utility Class

In all HARA utilities the local absolute risk tolerance coefficient is λw_0, whereas it is just λ for the exponential utility. Thus, HARA utility values with $w_0 = 1$ are comparable with exponential utility values, since the initial absolute risk tolerance is identical. Setting $w_0 = 1$, Figure 3 graphs the exponential, logarithmic, hyperbolic and power utility real option values as a function of λ, with $0 < \lambda \leq 1$, for the options to invest at market price and at a fixed strike equal to the current market price. The asset again has zero cash flows and the tree is again based on the parameters (20). The exponential utility yields the lowest and the logarithmic utility the highest option values.



(a) $\alpha = 0$ (b) $\alpha = 1$

Figure 3: Comparison of invest option values under exponential, logarithmic, power and hyperbolic utilities as a function of risk tolerance. Real option values on the vertical scale have been multiplied by 1000 for clarity. Parameters are as in (20).

For extremely high risk tolerance, i.e. $\lambda > 1$, hyperbolic utility values still lie between the exponential and logarithmic values, but the power utility values exceed even the logarithmic values, and as λ increases further the power values can become very large indeed, because the risk tolerance increases too rapidly with wealth.

However, HARA utilities are not always well-behaved. For instance, when $w < w_0$ the logarithmic local coefficient of risk tolerance is relatively small, and if the distribution of w is skewed to have mean below w_0 the logarithmic real option value can even be smaller than the exponential utility option value. Also, the logarithmic utility value becomes highly sensitive to small changes in w as it decreases towards the asymptote at the boundary $(1-\lambda)w_0$ because, in the normal range of $(0, 1)$ for λ, the logarithmic local risk tolerance is the most sensitive to wealth (recall, $\eta = 1$). Even the hyperbolic utility, with $\eta = 0.5$, can yield unstable results near the lower boundary. On the other hand, when $\lambda > 1$ the power utility (which has $\eta = \lambda$) has a risk tolerance that is much too sensitive to changes in wealth.

Exponential utility values are very stable; indeed they can yield analytic solutions in many cases. However, they have the CARA property which is widely regarded as being unrealistic. Indeed, exponential utility values for real options are typically too low because they ignore the possibility that the decision maker's risk tolerance could increase with wealth. If it does, the real option's value will always be greater than it is for an investor with constant (CARA) risk aversion.

We conclude that, of the four HARA utilities considered, power utilities produce the most reliable real option values for most reasonable values of risk tolerance, i.e. for $0 \leq \lambda \leq 1$, but for decision makers with very high risk tolerance logarithmic or hyperbolic utility representations would be more appropriate, the former giving real option values that are greater than the latter.

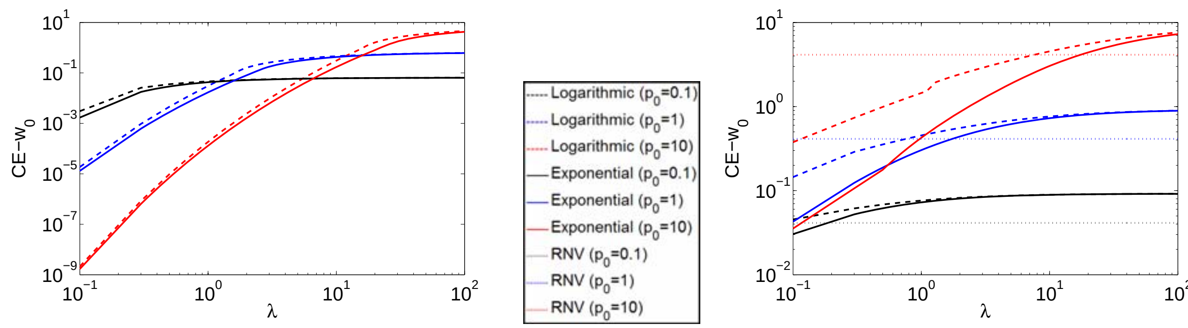
4.5 Relative Cost of the Asset

Here we examine how the current price of the asset affects terminal wealth if the asset is purchased now or in the future. When the price of the asset is high relative to initial wealth the terminal P&L becomes relatively more variable and this could increase or reduce the option's value, depending on the way that risk tolerance changes with wealth. The P&L arising from a low-cost asset would hardly change over the investment horizon, but it could change a lot for a high-cost asset with the same volatility. Thus, different optimal decisions and option values may be obtained for assets with a current price that is low or high, relative to initial wealth, even when the returns on the two assets have the same mean and variance.

We set the asset price at time 0 to be $p_0 = 0.1, 1$ and 10 and assume the investor's initial wealth is $w_0 = 1$ so that the investments have different sizes relative to the initial wealth of the investor. Suppose that the asset price trees have parameters:

$$T' = 5, \Delta t = 1/12, k = 3, r = 5\%, \mu = 15\%, \sigma = 50\%. \quad (20)$$

Figure 4 displays results for an exponential and a logarithmic utility for different values of the risk tolerance λ at the initial wealth of the investor. Both the option value and λ are represented on a base 10 log scale. Figures 4a and 4b compare the option values for $\alpha = 0$ (invest at market price) and $\alpha = 1$ (fixed ATM strike at p_0).



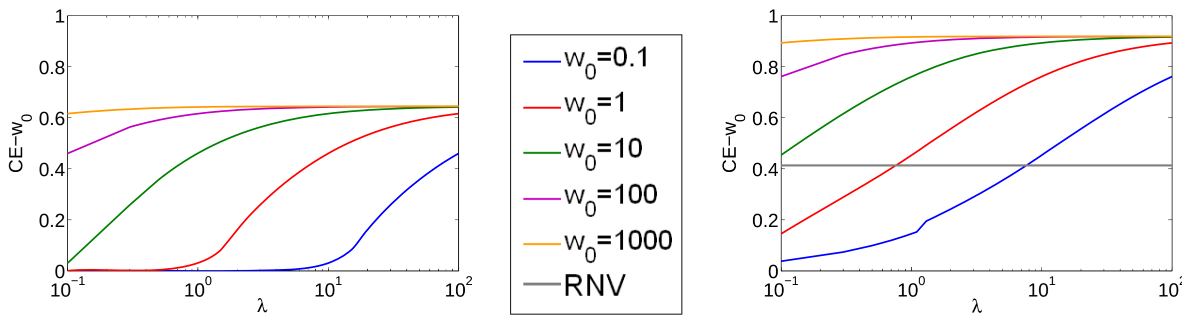
(a) $\alpha = 0$ (b) $\alpha = 1$

Figure 4: Real option values under exponential and logarithmic utilities as a function of risk tolerance λ, $T' = 5$, $\Delta t = 1/12$, $k = 3$, $r = \tilde{r} = 5\%$, $\mu = 15\%$, $\sigma = 50\%$, $K = p_0 = \$0.1, \$1, \$10$m, $w_0 = 1$m, $0.1 \leq \lambda \leq 100$, CE in \$m. Both axes are in $\log_{10}$ scale.

In each case option value converges to the value obtained for a risk-neutral investor as $\lambda \to \infty$, and for the fixed-strike ATM option this value increases with p_0. The option price under the standard assumption of a complete market is marked by the dotted lines in Figure 5b (and recall that the RNV price is always zero in Figure 5a). This is less than the risk-neutral subjective value because $\mu > r$, but it would be greater than the risk-neutral subjective value when $\mu < r$. Also, the exponential option values (solid lines) are always less than or equal to the values obtained under the logarithmic utility (dotted lines) with the same initial risk tolerance, especially for the fixed-strike option and low values for λ. The option values for the high-priced asset, represented by red lines, are most sensitive to λ and the values for the low-priced asset, represented by black lines, are least sensitive to λ. These results demonstrate that the smaller (greater) the risk tolerance of the investor, the higher he ranks the option to invest in the relatively low-priced (high-priced) asset, given that the asset-price dynamics follow the same GBM process.[17]

[17]For instance, an investor with initial wealth \$1m, an exponential utility and a risk tolerance of 0.5 would prefer the option to invest in the asset with current price \$0.1m; but if his risk tolerance was 5 he would prefer the option on the asset with current price \$1m; and if his risk tolerance were 50 he would prefer the option to invest in the asset with current price \$10m.

An alternative perspective is of several decision makers having different levels of initial wealth w_0 but the same relative risk tolerance at w_0, all valuing the same investment opportunity under identical views for the asset price process. Here we set the asset price at time 0 to be $p_0 = 1$ but the investment has different sizes relative to the different initial wealth of the investors. For instance, for an investor with $w_0 = 100$ the current asset price is only 1/100th of the investor's initial wealth. There is one asset price tree, which we again suppose has parameters (20) and now let the investors have $w_0 = 0.1, 1, 10, 100$ or 1000. Figures 5a and 5b compare the option values for $\alpha = 0$ (invest at market price) and $\alpha = 1$ (fixed strike at \$1m) based on a logarithmic utility (17). We again display the option value as a function of the initial coefficient of relative risk tolerance λ taking values from 0.1 to 100 on a base 10 log scale. Note that an investor with $w_0 = 1000$ has an option value that is almost a straight line at the risk-neutral option value because his initial absolute risk tolerance λw_0 is so high.



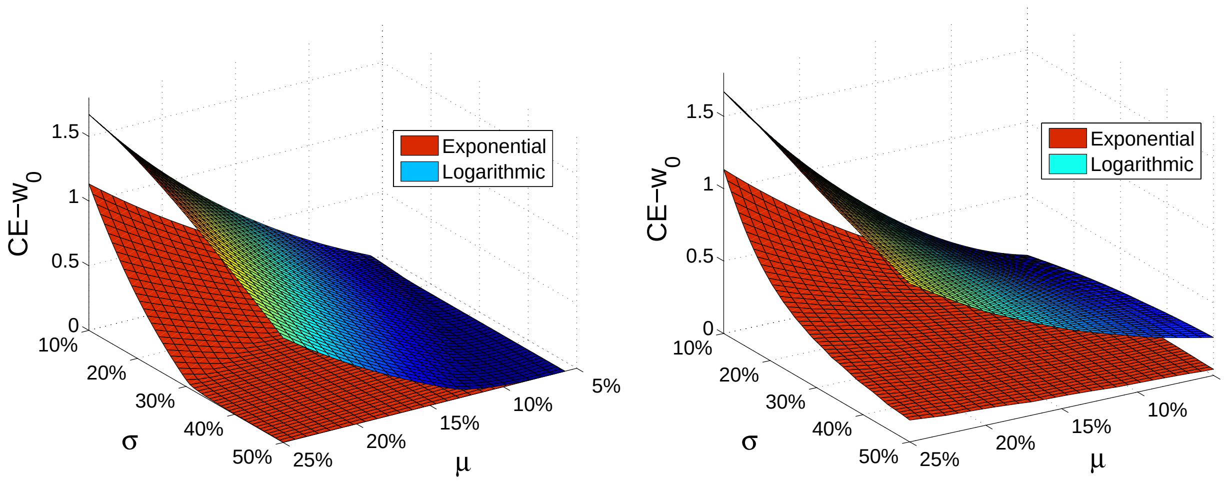
(a) $\alpha = 0$ (b) $\alpha = 1$

Figure 5: Comparison of option values under logarithmic utilities as a function of risk tolerance, $T' = 5$, $\Delta t = 1/12$, $k = 3$, $K = p_0 = \$1\text{m}$, $r = \tilde{r} = 5\%$, $\mu = 15\%$, $\sigma = 50\%$, $w_0 = \$0.1, 1, 10, 100, 1000\text{m}$, $0.1 \leq \lambda \leq 100$, CE in \$m. The λ axis is in $\log_1 0$ scale.

4.6 Sensitivity to μ and σ

While an investor's risk tolerance and its sensitivity to wealth may be defined fairly accurately using the techniques introduced by Keeney and Reiffa (1993), the values of μ and σ that underpin the real option valuation may be highly uncertain. This depends on the confidence that the decision maker has in his views about the price process: the more confident he is in his price forecasts the more certain he will be about the real option value. In this section we ask how sensitive real option values are to the subjective estimates for the drift and volatility of the price process.

Figure 6 depicts the values of a real option to invest as a function of μ and σ, under exponential and logarithmic utilities. Given our results in Section 4.4 these can be regarded as lower and upper bounds for the real option value. The other parameters are as stated in the legend to the figure. When $\alpha = 0$ the option value always decreases as uncertainty increases,



(a) $\alpha = 0$ (b) $\alpha = 1$

Figure 6: Value of an investment option under exponential and logarithmic utilities as a function of the investor's subjective views on expected return μ and volatility σ, for $\lambda = 0.2$. CE value in \$m for $p_0 = \$1\text{m}, w_0 = \$10\text{m}, r = 5\%, T' = 5, \Delta t = 1/12, k = 3$.

for any given expected return, due to the risk aversion of the decision maker. When there is high uncertainty (σ greater than about 30%) the exponential utility values are zero, i.e. the investment opportunity is valueless as the price would never fall far enough (in the decision maker's opinion) for investment to be profitable. By contrast, the logarithmic utility always yields a positive value provided the expected return is greater than about 10%, but again the risk-averse investor becomes more likely to defer investment as the volatility increases. Indeed, the option values are monotonically decreasing with σ for every μ and monotonically increasing with μ for every σ.

With low uncertainty and high expected return the two real options have similar values – as can be seen from the points above $(\mu, \sigma) = (25\%, 10\%)$ in the two surfaces. However, the fixed-strike option ($\alpha = 1$) has quite different sensitivity to μ and σ. This is particularly noticeable under the logarithmic utility, where the option values increase with volatility, except when the expected return is far above the risk-free rate. Of course, when $\mu = r$ (= 5% in this example) the option values increase monotonically with volatility, just as standard risk-neutral option prices do. Clearly then, the assumption about the investment costs affects not only the point valuation of a real option, but also the sensitivity of this value to changes in the investor's views about forward market prices.

The sensitivity of the option value to μ and σ also decreases as risk tolerance increases. To illustrate this we give a simple numerical example, for the case $\alpha = 0$ and an exponential utility. Suppose $\mu = 20\%$. If $\lambda = 0.2$ the option value is \$1,148 when $\sigma = 40\%$ and \$241,868 (almost 211 times larger) when $\sigma = 20\%$. When $\lambda = 0.8$ the option value is \$167,716 when $\sigma = 40\%$; but now when $\sigma = 20\%$ it only increases by a multiple of about 4, to \$713,812. Similarly, fixing $\sigma = 20\%$ but now decreasing μ from 20% to 10% the value changes from \$241,867 to \$109 (2,210 times smaller) for $\lambda = 0.2$, but from \$713,812 to \$113,959 (only about 6 times smaller) when $\lambda = 0.8$. Hence, the option value's sensitivities to μ and σ are much greater for low levels of risk tolerance.

5 Applications to Real Estate

This section applies our framework to decision problems encountered by private real-estate companies or housing trusts. The investor may be risk-neutral or risk-averse with a HARA utility, and where appropriate we compare the values obtained under the invest-at-market-price assumption with the values under a fixed-strike assumption (note that some authors, e.g. Howell et al. (2001) employ a fixed-cost assumption for real estate decision problems). First we examine the option values obtained when the decision maker's views are captured by the 'boom-bust' price process (3) and compare these with the values obtained under GBM views. Then show how the mean-rerting pruce assumption (4) may be employed in our framework, and compare the ranking attributed by different investors to two properties, characterized by different types of mean-reversion in their prices. After this we focus on the inclusion of cash flows, considering both positive cash flows for modelling buy-to-let real options and negative cash flows for modelling land development options, again showing how the ranking of different opportunities depends crucially on the assumptions made about the decision-maker's risk tolerance and its sensitivity to wealth.

5.1 Property Price Recessions and Booms

Many property markets are subject to bubbles and crashes. For example, based on monthly data on the Vanguard REIT exchanged traded fund (VQN), shown in Figure 5.1 the average annualised return from January 2005 to December 2006 June 2011 was 21% with a volatility of 15%. However, from January 2007 to December 2009 the property market crashed, with an average annualised return of -13% with a volatility of 58%, and from January 2010 to June 2011 the average annualised return was 22% with a volatility of 24%. Clearly, when the investment horizon is several years a property investor may wish to take account of both booms and busts in his views about expected returns, and we now give a numerical example of an investment option valuation under such scenarios.

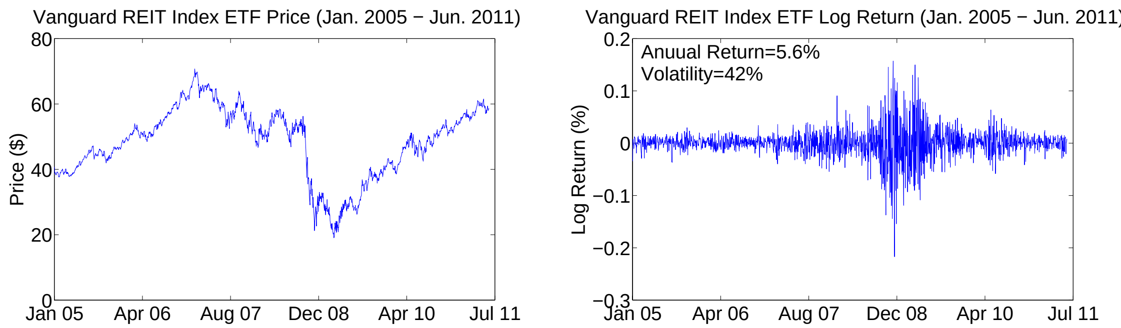


Figure 7: Monthly prices and returns on the Vanguard REIT exchanged traded fund, January 2005 to June 2011

Consider a simple boom-bust scenario over a 10 year horizon. The expected return is negative, $\mu_1 < 0$ for the first n years and positive, $\mu_2 > 0$ for the remaining $10 - n$ years. Following the above observations about VQN we set $\mu_1 = -10\%, \sigma_1 = 50\%$ and $\mu_2 = 10\%, \sigma_2 = 30\%$. We suppose decisions are taken every quarter with $\Delta t = 0.25$ and set $r = 5\%$ in the price evolution tree.

The real option values are given in Table 3, for investors having exponential utility and with varying levels of risk tolerance between 0.2 and 1. The property price recession lasts $n = 0, 2, 4, 6, 8$ or 10 years.[18] When $n = 0$ the investor expects the boom to last the entire period, but since $\sigma_2 = 30\%$ there is still uncertainty about the evolution of the market price, and with $\mu_2 = 10\%$ the price might still fall. The case $n = 10$ corresponds to the view that the market price to fall by $\mu_1 = -10\%$ each year for the entire 10 years, but with $\sigma_2 = 50\%$ this view is held with considerable uncertainty. At these two extremes, for any given λ the invest-at-market-price option has a lower value than the fixed-strike option because we have a standard GBM price process; this finding conforms with our earlier results. However, for intermediate values of n the invest-at-market-price option often has a higher value than the fixed-strike option. This ordering becomes more pronounced as n increases, because the investment cost decreases in the $\alpha = 0$ case and, provided that the price rises after the property is purchased, the profits would be greater than they are under the fixed-cost option.

This type of analysis would apply when an investor is considering which one of several properties to purchase, and the properties are in locations that have different market price characteristics. For instance, he may believe that there will be a few years of recession in the

[18] Results for the divest option, or for the invest option under different utilities are not presented, for brevity, but are available on request. The qualitative conclusions are similar.

$\alpha = 0$

λ \ n	0	2	4	6	8	10
0.2	866	382,918	**727,922**	438,482	174,637	0
0.8	211,457	1,230,035	**2,297,909**	1,148,695	370,354	0
1.0	266,752	1,349,295	**2,552,434**	1,300,724	406,567	0
∞	648,173	2,234,361	4,777,950	**5,308,672**	1,045,800	0

$\alpha = 1$

λ \ n	0	2	4	6	8	10
0.2	164,721	391,060	**552,578**	194,371	51,040	5,022
0.8	392,866	961,279	**1,704,007**	692,529	159,663	14,943
1.0	432,280	1,064,416	**1,938,157**	821,733	186,257	17,191
∞	740,493	1,989,551	4,339,659	**4,860,376**	841,706	50,787

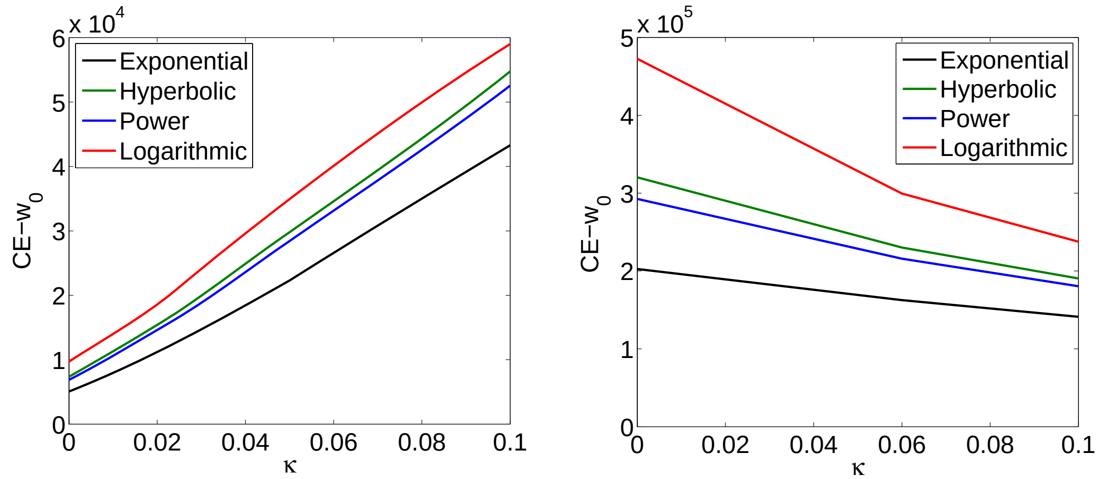
Table 3: Effect of a time-varying drift for the market price, with downward trending price for first n years followed by upward trending price for remaining $10-n$ years. Exponential utility with different levels of risk tolerance, with $\lambda = \infty$ corresponding to the risk-neutral (linear utility) value. Real option values in bold are the maximum values, for given λ. $p_0 = \$1$ million, $w_0 = \$1$ million, $T' = 10$, $\Delta t = 0.25$, $T = T' - \Delta t$, $r = 5\%$, $\mu_1 = -10\%$, $\mu_2 = 10\%$, $\sigma_1 = 50\%$ and $\sigma_2 = 20\%$.

UK and US housing markets, but that the US market will recover one or two years before the UK market. For risk-averse investors the maximum value arises when the length of the boom and bust periods are approximately the same. However, a risk-neutral investor would place the greatest value on a property location where the recession period lasts longer ($n = 6$ in our example).

5.2 Property Price Mean-Reversion

We now investigate how price mean-reversion influences real option values by considering a standard invest option with no cash flows when the market price follows the OU process (4). We set $\bar{p} = p_0$ for simplicity and employ the NR parameterization (5), allowing κ to vary between 0 and 0.1. Note that $\kappa = 0.1$ implies the fastest characteristic time to mean revert of 10 time-steps, so assuming these are quarterly this represents 2.5 years. Lower values for κ have slower mean-reversion, e.g. $\kappa = 0.02$ corresponds to a charachteristic time to mean revert of $\phi = 0.02^{-1}/4 = 12.5$ years if time-steps are quarterly. The other parameters are fixed, as stated in the legend to Figure 8, which displays the real option values for $\alpha = 0, 1$

as a function of κ.



(a) $\alpha = 0$ (b) $\alpha = 1$

Figure 8: Comparison of option values under HARA utilities with respect to mean-reversion rate κ. $w_0 = \$1$m, $r = 5\%$, $k = 1$ and $\lambda = 0.4$ $T' = 10$, $\Delta t = 1/4$, $K = p_0 = \$1$m, $T = T' - \Delta t$, $\sigma = 40\%$. Characteristic time to mean-revert $\phi = \Delta t/\kappa$ in years, e.g. with $\Delta t = 1/4$ then $\kappa = 0.02 \rightarrow \phi =$ 12.5yrs, $\kappa = 0.1 \rightarrow \phi = 2.5$yrs.

Under all of the utilities considered, the invest-at-market-price option values increase approximately linearly as the speed of mean-reversion speed increases. Recall that these option values increase with λ and μ but decrease with volatility and that increasing κ has the effect of decreasing the volatility of the terminal wealth distribution. In contrast, the fixed-strike real option values increase with volatility, and therefore they decrease as κ increases. We also find that, for fixed κ, the invest-at-market-price option value increases with λ and μ, as usual, but also it may now *in*creases with σ due to the positive effect of σ in the local drift (5d). However, this local drift effect is only evident for values of κ below a certain bound, which will depend on σ and on the utility function, λ and other parameters. For the parameter choice of Figure 8 the invest-at-market-price option values have their usual negative sensitivity to σ once κ exceeds approximately 2, where the characteristic time to mean revert is 1/8th of a year or less. Detailed results are not reported for lack of space, but are available from the authors on request.

Now consider an example in which different investors compare their valuation of two property investment opportunities, A and B. The investors have different utility functions and risk aversion but coincide on their views about the mean-reverting price processes governing each of the properties. Because the properties are in different locations, property A and B have different κ and σ. Again assume quarterly time-steps in the tree suppose that property A has a relatively rapid mean-reversion in its price ($\kappa = 1/10$, $\phi = 2.5$ years) with a low volatility ($\sigma = 20\%$) and property B has a relatively slow mean-reversion in its price ($\kappa = 1/40$, $\phi = 10$ years) with a higher volatility ($\sigma = 40\%$).

λ	0.4		0.8		∞	
	A	B	A	B	A	B
Exponential	**30,421**	12,952	**52,732**	43,947	98,077	**797,486**
Hyperbolic	**33,526**	17,563	55,230	**58,909**		
Power	**33,045**	16,651	56,365	**68,051**		
Logarithmic	19,890	**21,260**	56,986	**73,131**		

Table 4: A comparison of two invest option values for properties A and B, where $\kappa = 1/10$ and $\sigma = 20\%$ for property A whereas $\kappa = 1/40$ and $\sigma = 40\%$ for property B. We compare $\lambda = 0.4$ or 0.8 and all other parameter values are the same as in Figure 8. For each utility we highlight the preferred property.

Table 4 reports the values of the options to invest in each property for each investor, with the greater value of the two in bold. This shows that different investors would rank the two properties differently. A risk-neutral investor, or an investor with a logarithmic utility would always prefer property B, as would an investor with a relatively high risk tolerance ($\lambda = 0.8$) and a hyperbolic or a power utility.[19] All other investors would prefer property A. We conclude that the assumption made about the form of the utility function, as well as the level of risk tolerance for the decision maker, can have material consequences in the decision-making process.

5.3 Positive Cash Flows: Buy-to-Let Options

We now consider real options on properties that yield an income to the owner in the form of a buy-to-let residential property or office block, or a property such as a car park where fees accrue to its owner for its usage. Short-horizon decision trees for the invest and divest options on such a property are depicted in Figures 9 and 10. Rents, denoted $x_{\mathbf{s}(t)}$ in the trees, are captured using a positive dividend yield defined by (6) that may vary over time. Even a constant dividend yield would capture rents that increase/decrease in line with the market price. Rents are not re-invested in this property or in another property with the same

[19]Indeed, the risk-neutral investor places an extremely high value of \$797,486 on property B because he ignores the high uncertainty in the price of property B, and thinks only of the local drift which, with a high volatility but slow mean-reversion, could easily result in a substantial fall in the asset price followed by a rise before the investment horizon.

risk-return characteristics; otherwise we could employ the cum-dividend price approach that has previously been considered. Instead, cash flows are assumed to earn the risk-free rate, as to suppose they are invested in another risky project would introduce an additional source of uncertainty which is beyond the scope of this paper.

		Buy		Sell	
Utility \ Property	σ	40%	25%	30%	20%
	μ	15%	10%	15%	10%
	δ	10%	10%	20%	10%
Exponential		289	**316**	**6,775**	6,052
Hyperbolic		**348**	339	**6,610**	6,103
Power		**340**	336	**7,283**	6,296
Logarithmic		**358**	345	4,286	**5,330**

Table 5: Columns 2 and 3 compare the values of 2 real options, each to purchase a buy-to-let property based on the decision tree shown in Figure 9. Columns 4 and 5 compare the values of 2 real options, each to sell a buy-to-let property, based on the decision tree in Figure 10. For the decision maker, in each case $\lambda = 0.4$, $r = 5\%$, $w_0 = \$1$ million and for each property $p_0 = \$1$ million. The decision maker's beliefs about μ, σ and δ depend on the property's location. For each utility we highlight in bold the preferred location for buying (and selling) the property.

Each time a cash flow is paid the market price jumps down from p_t^+ to $p_t^- = p_t^+ - x_t$. Between payments the decision maker expects the discounted market price to grow at rate $\mu - r$, and based on the discretisation (2) we have $p_{t+1}^+ = up_t^-$ or $p_{t+1}^+ = dp_t^-$ with equal probability. The terminal nodes of the tree are associated with the increment in wealth $w - w_0$ where the final wealth w is given (8) for the option to invest in Figure 9, and by (10) for the option to divest in Figure 10, now setting CF$= x$.

We now apply the decision trees Figure 9 and Figure 10 to compare the values of the options to buy (or to sell) two buy-to-let properties in different locations: e.g. property A represents a typical office development in the UK and property B represents a similar, typical office development in the US. Both properties have current market value $p_0 = 1$, the initial wealth of the investor is $w_0 = 1$ and the risk-free rate $r = 5\%$. In each case rents are paid every six months, and are set at a constant percentage δ of the market price at the time the rent is paid. The investor has different views about the future market price and rents on each property, as specified in Table 5. The preferred property of the two to buy (or sell) has a value marked in bold in the table.

The invest option results, displayed in columns 2 and 3 of Table 5 demonstrate that a decision maker with exponential (CARA) utility would prefer the option to buy the second

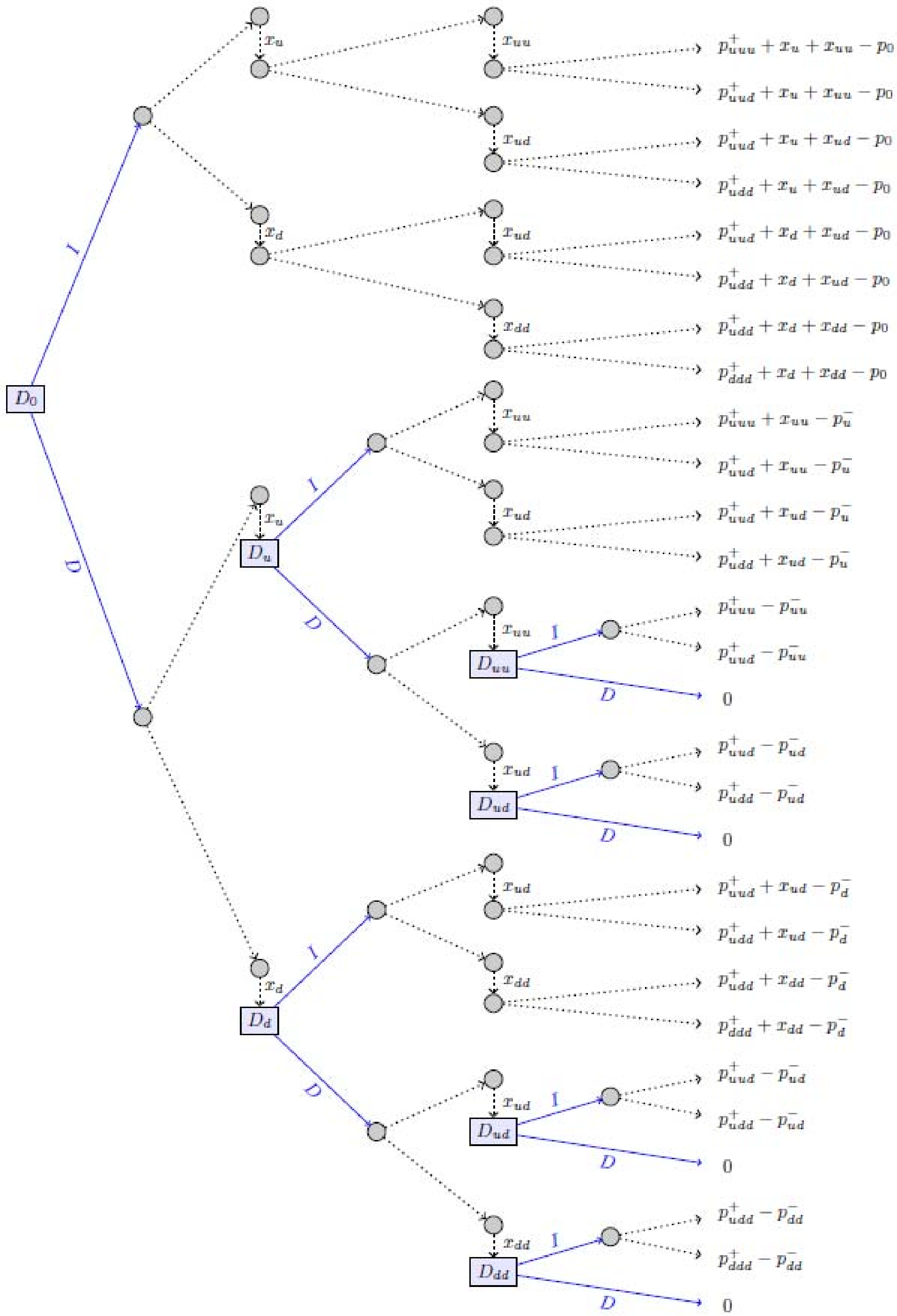


Figure 9: Decision tree depicting the option to invest in a property that pays rents, $x_{\mathbf{s}(t)}$. $T' = 3, T = 2, k = 1$. Terminal nodes labelled with P&L $(w - w_0)$, given by (8) following decision to invest (I) and 0 otherwise, with CF $= x$.

property, whilst a decision maker with any of the other HARA utilities would prefer the option to buy the first property. Similarly, considering columns 4 and 5, which state the divest real option values of two other properties, both currently owned by the decision maker, we see that all but the logarithmic utility would favour selling the first property.

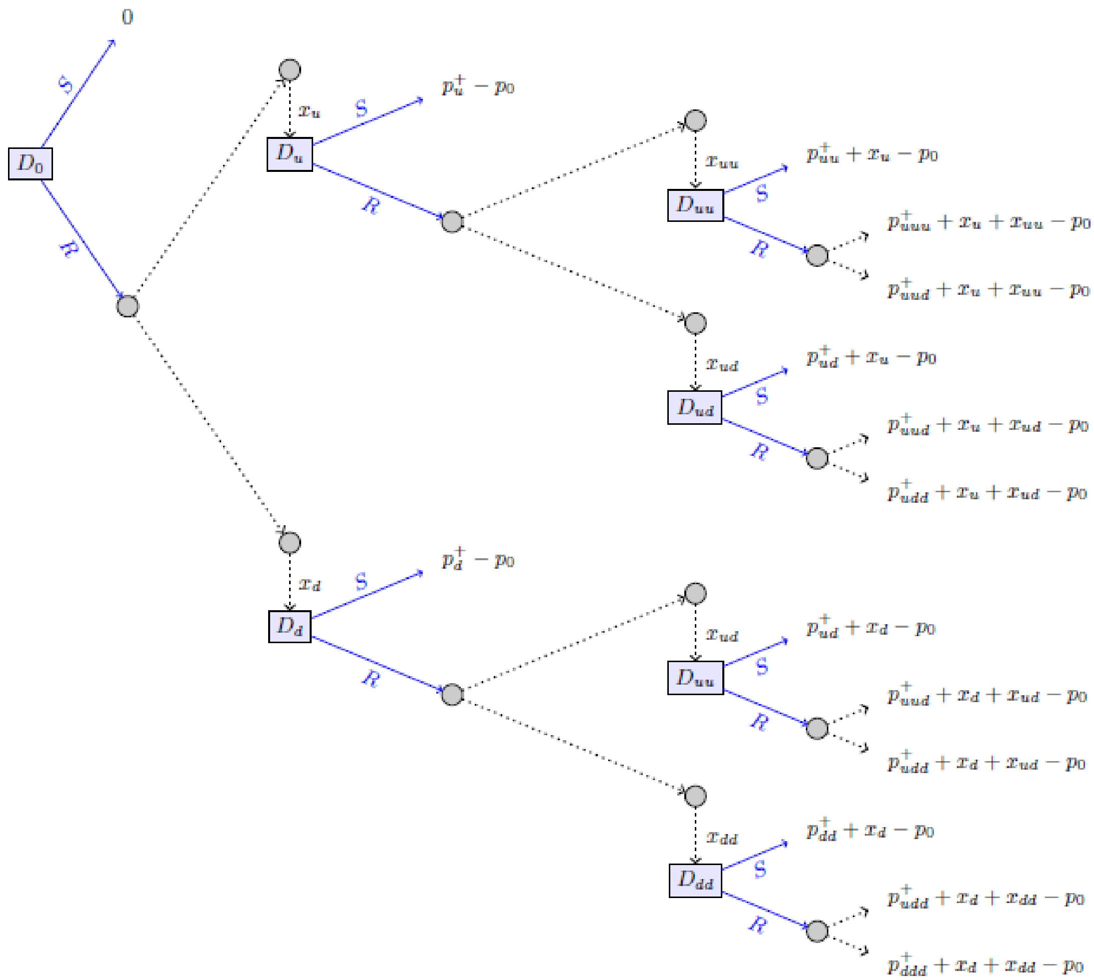


Figure 10: Decision tree depicting the option to sell a property that pays rents, $x_{\mathbf{s}(t)}$. $T' = 3, T = 2, k = 1$. Terminal nodes labelled with P&L $(w - w_0)$, given by (10) with CF $= x$ if the owner remains invested (R), or if the owner sells the property (S) by the difference between the selling price and initial price.

5.4 Negative Cash Flows: Buy-to-Develop Options

Setting a negative dividend yield is a straightforward way to capture cash that is paid into the land or property to cover development costs. But there are other important differences between the buy-to-develop and buy-to-rent option above. In the buy-to-develop case (a) there are no cash flows until the land or property is purchased; (b) these cash flows are included in the market price, so the market price following an investment decision is cum-dividend and prior to investment the market price evolves as in the zero cash flow case; and (c) the investment horizon T' is path dependent because it depends on the time of the investment (e.g. it takes 2 years to develop the property after purchasing the land). In contrast, for the buy-to-rent case above we have assumed (a) the property earns cash flows for whomsoever is the current owner; (b) the market price jumps down from p_t^+ to p_t^- at the time the rent is paid; and (c) the investment horizon is fixed at the time-length of the buying or selling opportunity (e.g. the property is expected to be on the market for 6 months).

A simple decision tree for the buy-to-develop option is depicted in Figure 11, in which the development cost is $y_{\mathbf{s}(t)} > 0$ and $\tilde{p}_t$ is the (cum-dividend) market price. The option maturity T is 2 periods, and so is the development time, so T' varies from 2 to 4 periods depending on the time of investment. To keep the tree simple we suppose that development costs are paid only once, after 1 period, to allow for planning time. For example, consider the node labelled D_u that arises if the investor does not purchase the land or property at $t = 0$ and subsequently the market price moves up at $t = 1$. A decision to invest at this time leads to four possible P&L's. For instance, following the dotted red lines, if the price moves up again at $t = 2$ the development cost at this time is y_{uu}, based on the market price of uup_0. But if the price subsequently moves down at $t = 3$, the terminal value of the property is $\tilde{p}_{1,uud} = d(uup_0 + y_{uu})$ and the costs are the sum of the price paid for the land and the development cost, i.e. $up_0 + y_{uu}$.

Utility \ Property		A	B
	σ	25%	15%
	μ	10%	35%
	δ	20%	40%
Exponential		**289**	182
Hyperbolic		299	**326**
Power		315	**350**
Logarithmic		**41**	0

Table 6: Value comparison of real options to buy two different properties for development based on decision tree shown in Figure 11. Here $\lambda = 0.4$, $r = 5\%$ and $w_0 = \$1$ million. Each property has $p_0 = \$1$ million but the decision maker's views on μ, σ and development costs δ differ for each property as shown in the table. The preferred option is indicated by the value in bold.

Table 6 displays some numerical results for the decision tree in Figure 11, reporting the value of two options to buy-to-develop land, each with initial market price \$1 million and $r = 5\%$, but the options have different μ, σ and development costs δ. For each option

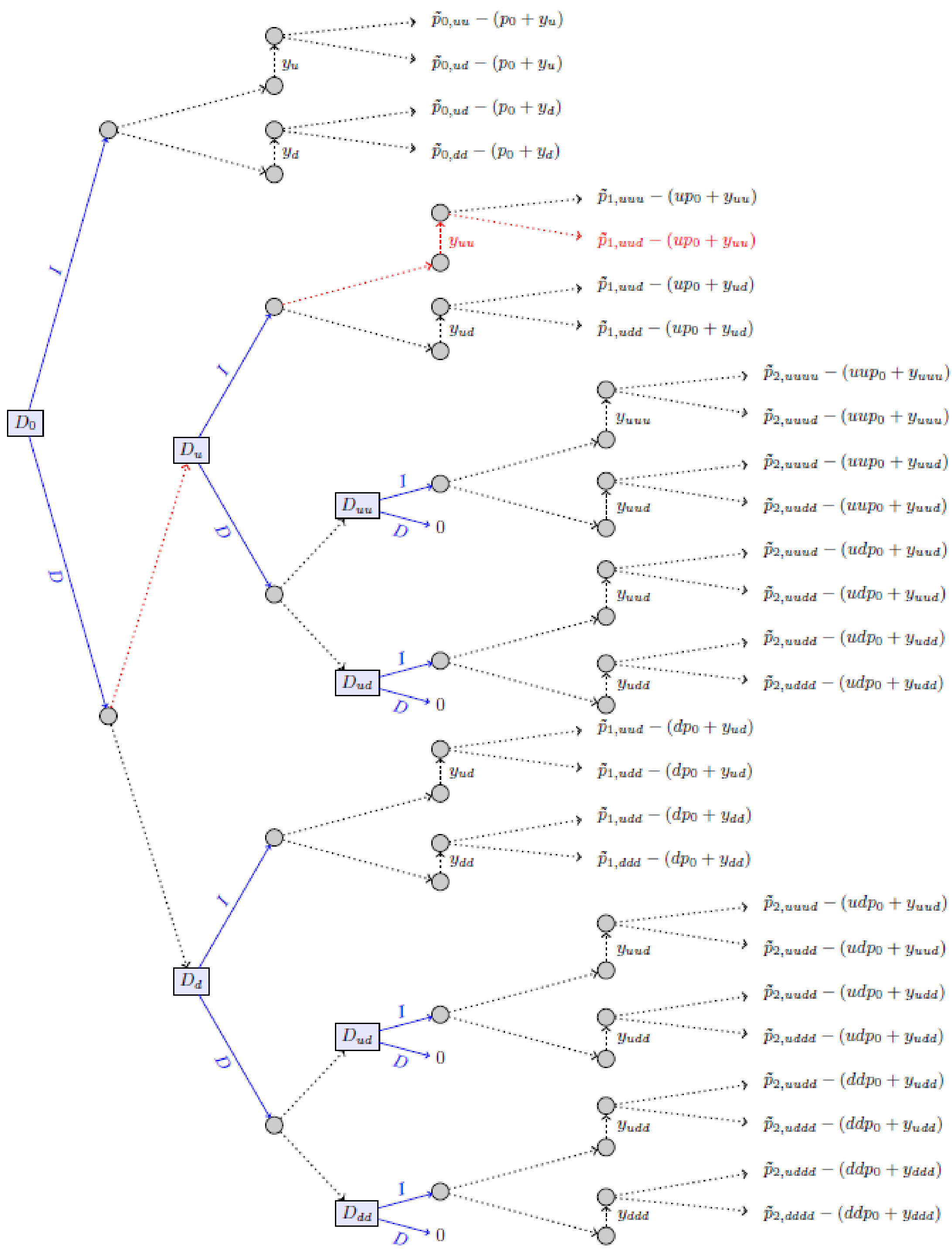


Figure 11: Decision tree depicting the option to invest in land for development. If the land is acquired (I) the development takes 2 periods and development costs occur only after the first period. Terminal nodes are associated with the P&L, $w - w_0$ resulting from the decision.

we suppose the development takes one year in total with the costs paid six months after purchase. The investors all have initial wealth \$10 million and $\lambda = 0.4$. In general, a higher development cost for given μ and σ decreases the buy-to-develop option value, and the investor becomes more likely to defer investment until the market price falls. But the option value also increases with μ and decreases with σ. We find that option A is preferred by an investor with an exponential or logarithmic utility whereas option B is preferred by an investor with a hyperbolic or a power utility. Hence, different investors that have identical wealth, share the same initial risk tolerance, and hold the same views about development costs and the evolution of market prices could still rank the values of two land-development options differently, just because their risk tolerance has different sensitivity to changes in wealth.

6 Summary and Conclusion

This paper introduces a general decision-tree framework for valuing real options that is flexible enough to encompass most real-world applications. In addition to the fixed-strike, complete markets assumption that is most commonly employed in the literature our approach encompasses real options to invest or divest at the market price of an asset in a market that need not be complete. We do not focus on analytic solutions and are therefore free to employ any representative utility function and any asset price process that we choose. We consider utilities in the HARA class and three possible asset price scenarios: the standard GBM, regime-switching GBM and a mean-reverting price process.

We provide many numerical illustrations to answer several important questions relating to real options that have not previously been addressed. Our main findings are as follows: (1) The assumption about the investment cost – whether it is fixed (in time 0 terms) or stochastic (and perfectly correlated with the market price) – has a significant influence on the real option value. The fixed-strike assumption can significantly over-estimate the value of a real option when the more-appropriate assumption is that the investment cost is at market price, or has both a fixed cost and market price component; (2) Under the GBM assumption the sensitivity of a fixed-strike real option to the volatility of the underlying asset price may be positive, whereas the invest-at-market-price real option always has a negative sensitivity to the asset-price volatility; (3) The value of the investment or divestment option increases with the frequency of decision opportunities. Therefore it is important to account for the flexibility of the decision-making process when valuing real options; (4) The decision maker's ranking of different real options depends on the form of the utility function applied, as well as his initial level of risk tolerance. Using HARA utilities, where the relative risk tolerance can change with wealth, gives a more accurate valuation of real options in incomplete markets than using an exponential utility – despite its tractability – because it has the unrealistic CARA property; (5) The price of the investment relative to the decision maker's wealth matters: the smaller (greater) the risk tolerance of the investor, the higher he ranks the option to invest in a relatively low-priced (high-priced) asset, given that the asset-price dynamics follow the same GBM process. Perhaps most importantly, we compare the standard RNV real option price with the values that would be ascribed to the option by risk-averse investors under the incomplete market assumption. Thus, we provide straightforward comparison between the real option price that is obtained using standard, but unrealistic, assumptions with the value that would be found using a more complete and general approach.

Our framework is sufficiently flexible to handle a variety of real options on real estate. Numerical results for different types of property investment and divestment decisions have been used to illustrate how our methodology can be implemented using mean-reverting or 'boom-bust' property price scenarios. We have also shown how the ranking of various real estate options, including buying or selling a property that pays rents and investing in a fixed-term land development, depends on the investor's initial risk tolerance and its sensitivity to wealth. This research has potential applications to many other types of real options and management decisions, not just to real estate problems. Also, the methodology could be further developed in several ways: to utility functions outside the HARA class, to more complex views on market prices than lognormal, mean-reverting or boom-bust scenarios, and to include stochastic financing costs and/or cash flows.

References

M. Benaroch and R. J. Kauffman. Justifying electronic banking network expansion using real options analysis. MIS Quarterly, 24:197–225, 2000.

P. F. Boer. Valuation of technology using "real options". Research Technology Management, 43:26–30, 2000.

E. H. Bowman and G. T. Moskowitz. Real options analysis and strategic decision making. Organization Science, 12:772–777, 2001.

L. E. Brandão and J. S. Dyer. Decision analysis and real options: A discrete time approach to real option valuation. Annals of Operations Research, 135(1):21–39, 2005. doi: 10.1007/s10479-005-6233-9.

L. E. Brandão, J. S. Dyer, and W. J. Hahn. Using binomial decision trees to solve real-option valuation problems. Decision Analysis, 2:69–88, 2005.

L. E. Brandão, J. S. Dyer, and W. J. Hahn. Response to comments on brandão et al. (2005). Decision Analysis, 2:103–109, 2008.

D. R. Capozza and G. A. Sick. Valuing long-term leases: the option to redevelop. Journal of Real Estate Economics and Finance, 4:209–233, 1991.

D. M. Chance. Convergence of the Binomial to the Black-Scholes Model. E. J. Ourso College of Business, Lousiana State University, 2163 Patrick F. Taylor Hall, Baton Rouge, LA 70803, US, 2008.

T. Copeland and V. Antikarov. Real Options: A Practitioner's Guide. Texere, New York, 2001.

T. Copeland, T. Koller, and J. Murrin. Valuation: Measuring and Managing the Value of Companies. New York: John Wiley & Sons, 1990.

J. C. Cox, S. A. Ross, and M. Rubinstein. Option pricing: A simplified approach. Journal of Financial Economics, 7(3):229–263, Sept. 1979.

M. H. A. Davis, Y. Kabanov, R. Lipster, and J. Stoyanov. Optimal hedging with basis risk. Stochastic Calculus to Mathematical Finance: The Shiryaev Festschrift, pages 169–187, 2006.

J. Evans, V. Henderson, and D. G. Hobson. Optimal timing for an indivisible asset sale. Mathematical Finance, 18(4):545–567, 2008.

M. R. Grasselli. Getting real with real options: A utility-based approach for finite-time investment in incomplete markets. Journal of Business Finance and Accounting, 38(5-6):740–764, 2011.

S. R. Grenadier. The strategic exercise of options: development cascades and overbuilding in real estate markets. The Journal of Finance, 51:1653–1679, 1996.

J. M. Harrison and D. M. Kreps. Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory, 20:381–408, 1981.

V. Henderson. Valuation of claims on nontraded assets using utility maximization. Mathematical Finance, 12(4):351–373, 2002.

V. Henderson. Valuing the option to invest in an incomplete market. Mathematics and Financial Economics, 1(2):103–128, 2007.

V. Henderson and D. G. Hobson. Real options with constant relative risk aversion. Journal of Economic Dynamics and Control, 27(2):329–355, 2002.

S. Howell, A. Stark, D. Newton, D. Paxson, M. Cavus, J. Pereira, and K. Patel. Real Options: Evaluating Corporate Investment Opportunities in a Dynamic World. Financial Times, Prentice Hall, 2001.

R. Jarrow and A. Rudd. Approximate option valuation for arbitrary stochastic processes. Journal of Financial Economics, 10:347–369, 1982.

E. Kasanen and L. Trigeorgis. Real Options in Capital Investment: Models, Strategies, and Applications, chapter Merging Finance Theory and Decision Analysis, pages 47–68. Westport, Conn,: Praeger, 1995.

R. Keeney and H. Reiffa. Decisions with Multiple Objectives. Cambridge University Press, New York, 1993. Original edition published by Wiley, New York, 1976.

B. Kogut. Joint ventures and the option to expand and acquire. Management Science, 37: 19–33, 1991.

S. P. Mason and R. C. Merton. Recent Advances in Corporate Finance, chapter The Role of Contingent Claims Analysis in Corporate Finance, pages 7–54. Irwin, Boston, MA, 1985.

R. McDonald and D. Siegel. The value of waiting to invest. Quarterly Journal of Economics, 101(4):707–727, 1986.

R. C. Merton. Lifetime portfolio selection under uncertainty: the continuous time case. Review of Economics and Statistics, 51:247–257, 1969.

R. C. Merton. Optimum consumption and portfolio rules in a continuous-time model. Journal of Economic Theory, 3:373–413, 1971.

J. Miao and N. Wang. Investment, consumption, and hedging under incomplete markets. Journal of Financial Economics, 86(3):608–642, 2007.

J. Mossin. Optimal multiperiod portfolio policies. Journal of Business, 41:215–229, 1968.

S. C. Myers. Determinantes of corporate borrowing. Journal of Financial Economics, 5(2): 147–175, 1977.

D. B. Nelson and K. Ramaswamy. Simple binomial processes as diffusion approximations in financial models. The Review of Financial Studies, 3:393–430, 1990.

S. Panayi and L. Trigeorgis. Multi-stage real options: the cases of information technology infrastructure and international bank expansion. The Quarterly Review of Economics and Finance, 38:675–682, 1998.

K. Patel, D. Paxson, and T. F. Sing. A review of the practical uses of real property options. RICS Research Paper Series, 5:1, 2005.

L. Quigg. Empirical testing of real option-pricing models. The Journal of Finance, 48:621–640, 1993.

M. B. Shackleton and R. Wojakowski. Finite maturity caps and floors on continuous flows. Journal of Economic Dynamics and Control, 31(12):3843–3859, 2007.

H. T. J. Smit. The valuation of offshore concessions in the netherlands. Financial Management, 26(2):5–17, 1996.

H. T. J. Smit and L. A. Ankum. A real option and game-theoretic approach to corporate investment strategy under competition. Financial Management, 22(3):241–250, 1993.

J. E. Smith. Alternative approaches for solving real-options problems. Deicison Analysis, 2 (2):89–102, 2005.

J. E. Smith and K. F. McCardle. Valuing oil properties: Integrating option pricing and decision analysis approaches. Operations Research, 46(2):198–217, 1998.

J. E. Smith and R. F. Nau. Valuing risky projects - option pricing theory and decision-analysis. Management Science, 41(5):795–816, 1995.

A. J. Triantis and J. E. Hodder. Valuing flexibility as a complex option. The Journal of Finance, 45:549–565, 1990.

L. Trigeorgis. The nature of option interactions and the valuation of investments with multiple real options. The Journal of Financial and Quantitative Analysis, 28:1–20, 1993.

L. Trigeorgis. Real Options: Managerial Flexibility and Strategy in Resource Allocation. The MIT Press, Cambridge, MA., 1996.

L. Trigeorgis and P. Mason. Valuing managerial flexibility. Midland Corporate Finance Journal, 5:14–21, 1987.

K. T. Yeo and F. Qiu. The value of management flexibility - a real option approach to investment evaluation. International Journal of Project Management, 21:243–250, 2002.

